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Adaptation of Sumudu Transform Iterative Method for Solving Fractional Integro-Differential Equations

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***Abstract*--In this paper, we discussed a new Sumudu transform iterative method and successfully applied on linear and nonlinear fractional integro-differential equations. The obtained results are compatible with those that got by other methods. A comparison between our method and other results is given. The fractional derivative here described in the Caputo sense. The proposed method has the power by reducing the size of calculations and by finding the exact solution.**

Index terms--New iterative Sumudu transform method; fractional integro-differential equations ; fractional Caputo derivative.

I. Introduction

Fractional calculus is considered a very important topic for many scientists due to the large number of applications in many fields such as engineering, physics and chemistry, economy, biology and so on; see [1,2,3,4,5]. Fractional integro-differential equations play an important role in different fields such as electromagnetic waves, biomedical engineering, fluid mechanics etc; for that studying fractional integro-differential equations has become a focus of interest for many researchers. Several methods have been employed efficiently to give best and accurate solutions for integro-differential equations such as Adomian decomposition method [6,7], Homotopy perturbation method [8], Fractional differential transformation method [9], Taylor expansion method [10], Legendre wavelet method [11], and Laplace variational iteration method [12].

In the literature several works were done on the concept of integral transform such as Fourier, Laplace, Melin, and Abel. Sumudu transform is one of the new integral transformation that has been introduced by [13], many properties have been presented at [14,15]. Sumudu transform is efficiently applied to many fractional partial, differential, and integro-differential equations [16-26].

The new Sumudu transform iterative method (NSTIM) has been successfully applied in many partial and ordinary differential equations [27-31]. In this paper, we prove the power of this method by applying it on many fractional integro-differential equations.

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II. Basic Definitions

In this section, some definitions of fractional calculus and Sumudu transform properties are presented.

Definition 1.

The Riemann-Liouville fractional integral operator I^{α} of order $\alpha > 0$, of a function $f \in C_{\mu}, \mu \geq -1$, is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$
$$I^0 f(t) = f(t)$$

Where Γ is the well-known Gamma function.

Some properties of I^{α} are

1. $I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t)$
2. $I^{\alpha} I^{\beta} f(t) = I^{\beta} I^{\alpha} f(t)$
3. $I^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$
4. $I^{\alpha} c = \frac{c}{\Gamma(\alpha+1)} t^{\alpha}$, c is constant.

Definition 2.

Caputo derivative of the function $f(t)$ is defined as

$$D^{\alpha} f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n-\alpha)}(\tau) d\tau, \quad t > 0$$

Where $n-1 < \alpha \leq n$, $n \in \mathrm{N}, f \in C_{-1}^{n}$

Some properties of Caputo derivative

1. $D^{\alpha} I^{\alpha} f(t) = f(t)$
2. $I^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}$

Definition 3.

If $f(t)$ is one of the following set of functions

$$A=\left\{f(t)/\exists M,\tau_1,\tau_2>0,|f(t)|<Me^{\frac{t}{\tau_j}},t \in (-1)^j\times[0,\infty),j=1,2\right\}$$

then the Sumudu transform of $f(t)$ is defined as

$$S[f(t)]=F(\omega)=\int_0^\infty f(\omega t)e^{-t}dt, \qquad \omega\in(\tau_1,\tau_2)$$

III. THE NEW SUMUDU TRANSFORM ITERATIVE METHOD (NSTIM)

In this section, we present the new Sumudu transform iterative method (NSTIM) which is a method that gives a solution of a convergent series form.

Consider the following initial value fractional integro-differential equation

$$D^\alpha y(t)=g(t)+h(t)y(t)+\int_0^t k(t,\tau)N(y(\tau))d\tau\,, \qquad t>0,\ n-1<\alpha\le n \tag{1}$$

Subject to the initial conditions $y^{(m)}(0)=a_m$, m= $0,1,\dots,n-1$,$n\in$ N

Where D^α is the Caputo fractional derivative of $y(t)$ and $N(y(t))$ is a linear or nonlinear continuous function of y , $a_k{}'s$ are constants and g, h, and k are given functions.
Now apply Sumudu transform on both sides of equation (1)

$$S[D^\alpha y(t)=S[g(t)]+S[h(t)y(t)+S[\int_0^t k(t,\tau)N(y(\tau))d\tau] \tag{2}$$

Apply Sumudu transform properties on equation (1) we get

$$\omega^{-\alpha}S[y(t)]-\sum_{k=0}^{n-1}\omega^{-\alpha+k}y^{(k)}(0)=S[g(t)]+S[h(t)y(t)]+S[\int_0^t k(t,\tau)N(y(\tau))d\tau] \tag{3}$$

By simplifying equation (3), it will be as

$$S[y(t)]=\sum_{k=0}^{n-1}\omega^k y^{(k)}(0)+\omega^\alpha S[g(t)]+\omega^\alpha S[h(t)y(t)]+\omega^\alpha\, S[\int_0^t k(t,\tau)N(y(\tau))d\tau] \tag{4}$$

Take the inverse Sumudu transform for both sides of equation (4), we have

$$y(t)=S^{-1}\left(\sum_{k=0}^{n-1}\omega^k y^{(k)}(0)\right)+S^{-1}(\omega^\alpha S[g(t)])+S^{-1}(\omega^\alpha S[h(t)y(t)])0+S^{-1}\left(\omega^\alpha S[\int_0^t k(t,\tau)N(y(\tau))d\tau]\right) \tag{5}$$

Now assume that

$$f(t)=S^{-1}\left(\sum_{k=0}^{n-1}\omega^k y^{(k)}(0)\right)+S^{-1}(\omega^\alpha S[g(t)])$$

$$L(y(t))=S^{-1}(\omega^\alpha S[h(t)y(t)])+S^{-1}\left(\omega^\alpha S[\int_0^t k(t,\tau)N(y(\tau))d\tau]\right) \tag{6}$$

Then the solution will be of the form

$$y(t)=f(t)+L(y(t)) \tag{7}$$

Now to find the solution assume that the solution has the series form

$$y(t)=\sum_{i=0}^\infty y_i(t) \tag{8}$$

If the function $L(y(t))$ is linear function, then it has the property

$$L(\sum_{i=0}^\infty y_i(t))=\sum_{i=0}^\infty L(y_i(t)) \tag{9}$$

The solution will be

$$y(t)=\sum_{i=0}^\infty y_i(t)=f(t)+L(\sum_{i=0}^\infty y_i(t))=f(t)+\sum_{i=0}^\infty L(y_i(t)) \tag{10}$$

And the iterations are

$$y_0=f(t) \tag{11}$$
$$y_1=L(y_0) \tag{12}$$
$$y_{n+1}=L(y_n),\ n\ge 1 \tag{13}$$

But if $L(y(t))$ is nonlinear, then it satisfies that

$$L(\sum_{i=0}^\infty y_i(t))=L(y_0)+\sum_{i=0}^\infty\left(L(\sum_{j=0}^i y_j)-L(\sum_{j=0}^{i-1}y_j)\right) \tag{14}$$

then the solution is

$$y(t)=\sum_{i=0}^\infty y_i(t)=f(t)+L(\sum_{i=0}^\infty y_i(t))=f(t)+L(y_0)+\sum_{i=0}^\infty\left(L(\sum_{j=0}^i y_j)-L(\sum_{j=0}^{i-1}y_j)\right) \tag{15}$$

And the iterations will be

$$y_0=f(t) \tag{16}$$
$$y_1=L(y_0) \tag{17}$$
$$y_{n+1}=L(\sum_{i=0}^n y_i(t))-L(\sum_{i=0}^{n-1}y_i(t)),\ n\ge 1 \tag{18}$$

Notice that in equation (1) if the function $g(t)$ is given as sum of many functions
$g(t)=g_1(t)+g_2(t)+\cdots$, then we can separate this function such as
$g_1(t)=f(t)$ and add the other functions $g_2(t)+g_3(t)+\dots$ to $L(y(t))$

IV. NUMERICAL APPLICATIONS

Now we apply the new Sumudu transform iterative method (NSTIM) on many linear and nonlinear integro-differential equations that have been solved in many other methods to show the accuracy of our method.

Example 1:
Consider the following linear Fredholm fractional integro-differential equation

$$\begin{cases} D^{\frac{1}{2}}y(t)=\frac{1}{\sqrt{\pi}}\left(\frac{8}{3}t^{\frac{3}{2}}-2t^{\frac{1}{2}}\right)+\frac{t}{12}+\int_0^1 t\tau y(\tau)d\tau,\ 0\le t,\tau\ge 1 \\ \quad \textit{subject to the initial condition } y(0)=0 \end{cases} \tag{19}$$

It was proved that the exact solution is $(t)=t^2-t$, see [32].

Now apply Sumudu transform for both sides of equation (19) and use Sumudu transform fractional derivative, we get

$$\omega^{-\frac{1}{2}}S[y]-\omega^{-\frac{1}{2}}y(0)=S\left[\frac{1}{\sqrt{\pi}}\left(\frac{8}{3}t^{\frac{3}{2}}-2t^{\frac{1}{2}}\right)\right]+S\left[\frac{t}{12}+\int_0^1 t\tau y(\tau)d\tau\right] \quad (20)$$

And by using initial condition it will be equivalent to

$$S[y]=\omega^{\frac{1}{2}}S\left[\frac{1}{\sqrt{\pi}}\left(\frac{8}{3}t^{\frac{3}{2}}-2t^{\frac{1}{2}}\right)\right]+\omega^{\frac{1}{2}}S\left[\frac{t}{12}+\int_0^1 t\tau y(\tau)d\tau\right] \quad (21)$$

Use inverse Sumudu transform, we obtain

$$y(t)=S^{-1}\left(\omega^{\frac{1}{2}}S\left[\frac{1}{\sqrt{\pi}}\left(\frac{8}{3}t^{\frac{3}{2}}-2t^{\frac{1}{2}}\right)\right]\right)+S^{-1}\left(\omega^{\frac{1}{2}}S\left[\frac{t}{12}+\int_0^1 t\tau y(\tau)d\tau\right]\right) \quad (22)$$

We need to use some properties of Gamma function such as

$$\Gamma(z+1)=z\Gamma(z), z>0, and\ \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi} \quad (23)$$

Use Sumudu transform properties in Table 1 and some properties of Gamma function in (23), equation (19) will be equivalent to

$$y(t)=S^{-1}(2\omega^2-\omega)+S^{-1}\left(\frac{\omega^{\frac{3}{2}}}{12}+\omega^{\frac{1}{2}}S\left[\frac{t}{12}+\int_0^t t\tau y(\tau)d\tau\right]\right) \quad (24)$$

Now according to the NSTIM

$y_0=S^{-1}(2\omega^2-\omega)=t^2-t$

and

$$y_{n+1}=S^{-1}\left(\frac{\omega^{\frac{3}{2}}}{12}+\omega^{\frac{1}{2}}S\left[\frac{t}{12}+\int_0^1 t\tau y_n(\tau)dd\tau\right]\right),\ n\geq 0 \quad (25)$$

Hence, we have

$$y_1=S^{-1}\left(\frac{\omega^{\frac{3}{2}}}{12}+\omega^{\frac{1}{2}}S\left[t\int_0^1 \tau^3-\tau^2 d\tau\right]\right)=S^{-1}\left(\frac{\omega^{\frac{3}{2}}}{12}+\omega^{\frac{1}{2}}S\left[\frac{-1}{12}t\right]\right)=S^{-1}\left(\frac{\omega^{\frac{3}{2}}}{12}-\frac{\omega^{\frac{1}{2}}}{12}\right)=0 \quad (26)$$

And then using equation (13) the other iterations will be zeros, $y_n=0, n\geq 1$.

Then the exact solution of equation (19) is $y(t)=\sum_{i=0}^{\infty}y_i(t)=t^2-t$ which is compatible with the exact solution that was found by other method in [32].

Example 2:

Consider the following nonlinear Volterra fractional integro-differential equation

$$D^{\frac{6}{5}}y(t)=\frac{5}{2\Gamma\left(\frac{4}{5}\right)}t^{\frac{4}{5}}-\frac{1}{252}t^9+\int_0^t (t-\tau)^2y^3(\tau)d\tau,\quad 0\leq t<1 \quad (27)$$

And subject to initial conditions $y(0)=y'(0)=0$

The exact solution was proved using other methods in [33] as y $(t)=t^2$.

The same solution was found using NSTIM

By taking the Sumudu transform for both sides of equation (33) we have

$$\omega^{-\frac{6}{5}}S[y]-\omega^{-\frac{6}{5}}y(0)-\omega^{-\frac{1}{5}}y'(0)=\frac{5}{2\Gamma\left(\frac{4}{5}\right)}\omega^{\frac{4}{5}}\Gamma\left(\frac{9}{5}\right)-\frac{\omega^9\Gamma(10)}{252}+\omega S[t^2]S[y^3]$$

By using initial conditions and taking Sumudu inverse for the equation we obtain

$$y(t)=S^{-1}\left(\omega^{\frac{6}{5}}\frac{5}{2\Gamma\left(\frac{4}{5}\right)}\omega^{\frac{4}{5}}\Gamma\left(\frac{9}{5}\right)\right)-S^{-1}\left(\omega^{\frac{6}{5}}\frac{\omega^9\Gamma(10)}{252}\right)+S^{-1}\left(\omega^{\frac{6}{5}}\omega S[t^2]S[y^3]\right)$$

Using the series form for $(t)=\sum_{i=0}^{\infty}y_i(t)$, the iterations will be

$$y_0=S^{-1}\left(\omega^{\frac{6}{5}}\frac{5}{2\Gamma\left(\frac{4}{5}\right)}\omega^{\frac{4}{5}}\Gamma\left(\frac{9}{5}\right)\right)=S^{-1}(2\omega^2)=t^2$$

And

$$y_{n+1}=-S^{-1}\left(\omega^{\frac{6}{5}}\frac{\omega^9\Gamma(10)}{252}\right)+S^{-1}\left(\omega^{\frac{6}{5}}\omega S[t^2]S[y_n^3]\right),\ n\geq 1$$

Then

$$y_1=-S^{-1}\left(\omega^{\frac{51}{5}}\frac{\Gamma(10)}{252}\right)+S^{-1}\left(\omega^{\frac{11}{5}}S[t^2]S[y_0^3]\right)$$
$$=-S^{-1}\left(\omega^{\frac{51}{5}}\frac{\Gamma(10)}{252}\right)+S^{-1}\left(\omega^{\frac{11}{5}}\omega^2\Gamma(3)\omega^6\Gamma(7)\right)$$

By using properties of the Gamma function $\Gamma(z+1)=z\Gamma(z)$ gives us $y_1=0$

According to equations (1) and (6) the function $L(y(t))$ is not linear

Hence $y_2=L(y_0+y_1)-L(y_0)=0$

Then other iterations of y are zeros $y_n=0, n\geq 1$

Finally, the solution of equation (27) is $y(t)=t^2$ is compatible with the same solution found in [33].

Example 3:

We finally consider the following linear Volterra fractional integro-differential equation that was studied in [9,12].

$$D^{\alpha}y(t)=t+(3+t)e^t+y-\int_0^t y(\tau)d\tau,\quad 0<t<1,\ 3<\alpha\leq 4 \quad (28)$$

with the initial conditions $y(0)=1, y'(0)=1, y''(0)=2, y'''(0)=3$

Now apply Sumudu transform for both sides of equation (28) and use Sumudu property of fractional derivative, we get

$$\omega^{-\alpha}(S[y]-y(0)-\omega y'(0)-\omega^2y''(0)-\omega^3y'''(0))=S\left[t+(3+t)e^t+y-\int_0^t y(\tau)d\tau\right]$$

By using Sumudu properties, it will be equivalent to

$$y=S^{-1}(1+\omega+2\omega^2+3\omega^3)+S^{-1}\left(\omega^{\alpha}S\left[t+(3+t)e^t+y-\int_0^t y(\tau)d\tau\right]\right)$$

Using NSTIM, we get the first iteration to be

$$y_0=S^{-1}(1+\omega+2\omega^2+3\omega^3)=1+t+t^2+\frac{t^3}{2}$$

Moreover, the next iterations are

$$y_{n+1}=S^{-1}\left(\omega^{\alpha}S\left[t+(3+t)e^t+y_n+\int_0^t y_n(\tau)d\tau\right]\right),\ n\geq 0$$

Now take a series approximation of exponential to be

$$e^t\approx 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}$$

The second iteration will be

$$y_1=S^{-1}(4\omega^{\alpha}+5\omega^{\alpha+1}+6\omega^{\alpha+2}+7\omega^{\alpha+3}+\omega^{\alpha+4})$$
$$=\frac{4t^{\alpha}}{\Gamma(\alpha+1)}+\frac{5t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{6t^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{7t^{\alpha+3}}{\Gamma(\alpha+4)}+\frac{t^{\alpha+4}}{\Gamma(\alpha+5)}$$

The solution with the first two iterations is

$$y(t) = y_0 + y_1 = 1 + t + t^2 + \frac{t^3}{2} + \frac{4t^{\alpha}}{\Gamma(\alpha+1)} + \frac{5t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{6t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{7t^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)}$$

The exact solution for equation (28) is $y = 1 + te^t$ for $\alpha = 4$. The numerical results in Table 1 show similar agreement with the results in [12].

These results were found using Mathematica package.

Table 1. Numerical results of Example 6 for $\alpha = 4$.

t	exact	Appr Sol $\alpha = 4$	Abs.Error	Rela. Error
0.	1.	1.	0.	0.
0.1	1.1105170918	1.1105170918	1.00342×10^{-12}	9.03561×10^{-13}
0.2	1.2442805516	1.2442805514	2.599×10^{-10}	2.08876×10^{-10}
0.3	1.4049576423	1.4049576355	6.74267×10^{-9}	4.7992×10^{-9}
0.4	1.5967298791	1.5967298109	6.82064×10^{-8}	4.27163×10^{-8}
0.5	1.8243606354	1.8243602235	4.11886×10^{-7}	2.2577×10^{-7}
0.6	2.0932712802	2.0932694851	1.79509×10^{-6}	8.57553×10^{-7}
0.7	2.4096268952	2.4096206477	6.24752×10^{-6}	2.59273×10^{-6}
0.8	2.7804327428	2.7804142978	1.84449×10^{-5}	6.63384×10^{-6}
0.9	3.2136428	3.2135947695	4.80306×10^{-5}	1.49458×10^{-5}
1.	3.7182818285	3.7181685406	1.13288×10^{-4}	3.04678×10^{-5}

V. CONCLUSIONS

In this paper, we successfully applied NSTIM to linear and nonlinear Volterra and Fredholm integro-differential equations. We compared our results with other numerical methods and got very good conclusions by finding an exact solution for linear integro-differential equations and good accurate solution for nonlinear integro-differential equation with quickly calculations. These comparisons show that our method is powerful and reliable.

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REFERENCES

[1] K. Oldham, and J. Spanier The fractional calculus theory and applications of differentiation and integration to arbitrary order, Elsevier, (1974).

[2] K.S. Miller, and B. Ross. "An Introduction to The Fractional Calculus and Fractional Differential Equations, John-Wily and Sons." Inc. New York (1993).

[3] I. Podlubny, Fractional Differential Equations, vol. 198 of Mathematics in Science and Engineering, Academic Press, San Diego, (1999).

[4] A. A. Kilbas, H. M. Srivastava, & J. J. Trujillo, Theory and applications of fractional differential equations (Vol. 204). Elsevier, (2006).

[5] K. Diethelm, N. J Ford, Analysis of fractional differential equations. Journal of Mathematical Analysis and Applications, 265(2), pp.229-248, (2002).

[6] S.A. Ahmed, T.M. Elzaki, On the comparative study integro–Differential equations using difference numerical methods. Journal of King Saud University-Science, 32(1), 84-89, (2020).

[7] S. Momani, M. A. Noor, Numerical methods for fourth-order fractional integro-differential equations. Applied Mathematics and Computation, 182(1), pp.754-760, (2006).

[8] P. Das, S. Rana, H. Ramos, Homotopy perturbation method for solving Caputo-type fractional-order Volterra-Fredholm integro-differential equations. Computational and Mathematical Methods, 1(5), e1047, (2019).

[9] A. Arikoglu and I. OzkolSolution of fractional integro-differential equations by using fractional differential transform method. Chaos, Solitons & Fractals, 40(2), 521-529, (2009).

[10] L. Huang, X.F. Li, Y. Zhao, X.Y Duan, Approximate solution of fractional integro-differential equations by Taylor expansion method. Computers & Mathematics with Applications, 62(3), 1127-1134, (2011).

[11] E. A Rawashdeh, Legendre wavelets method for fractional integro-differential equations. Applied Mathematical Sciences, 5(2), 2467-2474, (2011).

[12] T.A. Biala, Y.O Afolabi, O.O Asim, : Laplace variational iteration method for integro- differential equations for fractional order. International Journal of Pure and Applied Mathematics. 95, 413-426 (2014).

[13] G. K. Watugala, : Sumudu transform: a new integral transform to solve differential equations and control engineering problems. International Journal of Mathematical Education in Science and Technology. 24, 35–43 (1993).

[14] F. B. M. Belgacem, A. A. Karaballi,: Sumudu transform fundamental properties investigations and applications. Hindawi Publishing Corporation Journal of Applied Mathematics and Stochstic Analysis. (2006). https//doi.org/10.1155/JAMSA/2006/91083.

[15] F. B. M. Belgacem, A. A. Karaballi, S. L. Kalla,, Analytical investigations of the Sumudu transform and applications to integral production equations. Mathematical Problems in Engineering. 3, 103–118 (2003).

[16] F. Kaya, Y. Yılmaz, : Basic Properties of Sumudu Transformation and Its Application to Some Partial differential equations. Journal of Science. 23, 509-514 (2019).

[17] Z. Al-zhour, F. Alrawajeh, N. Al-mutairi, R., Alkhasawneh, : New results on the conformable fractional

Sumudu transform: Theories and Applications. International Journal of Analysis and Applications. 17, 1019-1033 (2019).

[18] A. Atangana, A. KJlJçman, : The Use of Sumudu Transform for Solving Certain Nonlinear Fractional Heat-Like Equations, Abstract and Applied Analysis. (2013).

[19] Y. A Amer, A. M. S. Mahdy, E. S. M. Youssef, : Solving Systems of Fractional Differential Equations Using Sumudu Transform Method. Asian Research Journal of Mathematics. 7, 1-15 (2017).

[20] S. T. Demiray, H. Bulut, , F. B. M. Belgacem, : Sumudu Transform Method for Analytical Solutions of Fractional Type Ordinary Differential Equations, Mathematical Problems in Engineering. (2015).

[21] R. Jain, D. Singh, : An Integro-Differential Equation of Volterra Type With Sumudu transform, Mathematica Aeterna. 2, 541 – 547 (2012).

[22] Eltayeb, H., Kılıçman, A., & Mesloub, S. Application of Sumudu decomposition method to solve nonlinear system Volterra integrodifferential equations. In Abstract and Applied Analysis (Vol. 2014), Hindawi, (2014).

[23] S.A. Ahmed, T.M. Elzaki, On the Comparative Study Integro – Differential Equations Using Difference Numerical Methods, Journal of King Saud University - Science (2018).

[24] T.M. Elzaki, and M., Chamekh, Solving nonlinear fractional differential equations using a new decomp osition method. Univ J Appl Math Comput, 6, pp.27-35, (2018).

[25] M.Z. Mohamed and T. M Elzaki, Comparison between the Laplace Decomposition Method and Adomian Decomposition in Time-Space Fractional Nonlinear Fractional Differential Equations. Applied Mathematics,9,448-458,(2018).

[26] T. M. Elzaki, Y. Daoud, & J. Biazar, Decomp osition method for fractional partial differential equations using modified integral transform. World Applied Sciences Journal, 37(1), 18-24, (2019).

[27] M. Kumar, V., Daftardar-Gejji, : Exact Solutions of Fractional Partial Differential Equations by Sumudu Transform Iterative Method. Fractional Calculus and Fractional differential Equations. (2019).

[28] K. Wang, S., Liu, : A new Sumudu transform iterative method for time-fractional Cauchy reaction–diffusion equation, Springer Plus. 5(1), 865 (2016).

[29] A. A. Hemeda, :New Iterative Method: Application to nth-Order Integro-Differential Equations, International Mathematical Forum. 7, 2317 – 2332(2012).

[30] A. Prakash, M. Kumar, D. Baleanu, : A new iterative technique for a fractional model of nonlinear Zakharov–Kuznetsov equations via Sumudu transform. Applied Mathhmatics and Computations. 334, 30-40 (2018).

[31] H. Anaç, M. Merdan, T. Kesemen, : Solving for the random component time-fractional partial differential equations with the new Sumudu transform iterative method, SN Applied Sciences. (2020).

[32] D. Sh. Mohammed, : Numerical Solution of Fractional Integro-Differential Equations by Least Squares Method and Shifted Chebyshev Polynomial. Mathematical Problems in Engineering. (2014).

[33] Y. Wang, L. Zhu, : Solving nonlinear Volterra integro-differential equations of fractional order by using Euler wavelet method, Advances in Difference Equations. (2017).

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