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Solving First-Order Nonlinear Fuzzy Initial Value Problems Using Two-Step Block Method with Presence of Higher Derivatives

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***Abstract*− Fuzzy differential equation models are suitable where uncertainty exists for real-world phenomena. Numerical techniques are used to provide an approximate solution to these models in the absence of an exact solution. However, existing studies that have developed numerical techniques for solving FIVPs possess an absolute error accuracy that could be improved. This is as a result of the low order and non-self-starting properties of the developed numerical techniques by previous studies. For this reason, this study, develops an Obrechkoff-type two-step implicit block method with the presence of second and third derivative for the numerical solution of first-order nonlinear fuzzy initial value problems. The convergence properties for the proposed block method are described in detail. Then the proposed method is adopted to solve first-order nonlinear fuzzy initial value problems with triangular and trapezoidal fuzzy numbers. The obtained results indicates that the proposed method effectively solves first-order nonlinear fuzzy initial value problems with better accuracy.**

***Index Terms*−Fuzzy Initial Value Problem, First-Order, Nonlinear, Obrechkoff-type, Two-Step, Block Method, Second derivative, Third derivative**

I. INTRODUCTION

Fuzzy differential equations (FDEs) are a powerful tool for analysing uncertainty in mathematical models, and this is evident in the studies [1-2] who introduced the fuzzy derivative concept. Seikkala [3] defined the first-order FIVP as the form,

$$y'(t) = f\big(t, y(t)\big)$$
$$y(t_0) = y_0, \quad t \in [t_0, T] \tag{1}$$

and various researchers have ventured in obtaining solutions to Equation (1) such as [4-7]. However, due to the complexity of the FDEs and presence of nonlinear terms, the exact solutions in certain cases are difficult and tedious to obtain. Therefore, numerical methods are introduced as an approximate approach to solving the FDEs. Many researchers developed numerical methods for the solution of Equation (1), and some recent studies include developed the Euler method [8-10]. The predictor–corrector technique was explored in [11-13]. All studies aimed to obtain better accuracy for the solution of these equations. For this purpose, developed block methods for first-order FIVPs [14-17]. The major drawbacks of these numerical approaches include having low order and not being self-starting but rather being implemented in predictor-corrector mode, which leads to computational complexity burden and low absolute error accuracy.

One of the numerical techniques for obtaining approximate solutions for differential equations with good accuracy is the block method [18-23]. For this reason, this study develops block methods to solve the first order nonlinear FIVPs considered in this article. A novel feature was introduced to the block method in this article by developing the method as Obrechkoff-type. The Obrechkoff techniques are a unique set of methods for numerically approximating differential equations. The existence of higher derivatives in the approach distinguishes this family of methods [24]. Thus, to handle the shortcoming of existing studies, this article proposes a self-starting Obrechkoff-type two-step implicit block method with the presence of second and third derivative with higher order compared to conventional two-step block methods. The convergence properties of the proposed method using definition of consistency and stability of the block methods are investigated, and then the proposed method is applied to solve some nonlinear

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FIVPS with initial conditions defined as both triangular and trapezoidal fuzzy numbers. The next section of this article defines some basic fuzzy theory preliminaries which are important to concept understanding. The third section of this article will show how the proposed method is derived while the next section discusses some basic properties of the block method. The fifth section will present the numerical examples and their results, and the last section conclude this work.

II. PRELIMINARIES

This section recalls some basic definitions which will be adopted in this article.

Definition 1: Triangular fuzzy number [25]

Consider that three numbers $(a,b,c) \in \mathbb{R}^3, a \leq b \leq c$. Then the triangular fuzzy number, $\psi(t)$ is given as

$$\psi(t;a,b,c) = \begin{cases} 0 & if\ t < a \\ \dfrac{t-a}{b-a} & if\ a \leq t \leq b \\ \dfrac{c-t}{c-b} & if\ b < t \leq c \\ 0 & if\ t > c \end{cases} \quad (2)$$

The corresponding $\alpha - level$ set of triangular fuzzy numbers is denoted as

$$[\psi]^\alpha = [a + \alpha(b-a), c - \alpha(c-b)], \quad (3)$$

$$\alpha \in [0,1]$$

Definition 2: Trapezoidal fuzzy numbers [25]

Consider that four numbers $(a,b,c,d) \in \mathbb{R}^4, a \leq b \leq c \leq d$, then the trapezoidal fuzzy number $\psi(t)$ is given as

$$\psi(t;a,b,c,d) = \begin{cases} 0 & if\ t < a \\ \dfrac{t-a}{b-a} & if\ a \leq t < b \\ 1 & if\ b \leq t \leq c \\ \dfrac{d-t}{d-c} & if\ c < t \leq d \\ 0 & if\ t > d \end{cases} \quad (4)$$

The corresponding $\alpha - level$ set of triangular fuzzy numbers is denoted as

$$[\psi]^\alpha = [a + \alpha(b-a), d - \alpha(d-c)], \quad (5)$$

$$\alpha \in [0,1]$$

Definition 3: Support of a Fuzzy Set [25]

Support of a fuzzy set $\hat{A}$ within the universal set T is defined as,

$$Supp(\hat{A}) = \{t \in T | \mu_{\hat{A}}(t) > 0\} \quad (6)$$

It contains all elements in T which degree of membership of fuzzy element is greater than zero.

Definition 4: α-Level (α-cut) [26]

Consider that $\psi \in R_{\tilde{f}}, \alpha \in [0,1]$, define the $\alpha-$ level set of fuzzy number denoted by $[\psi]^\alpha$ as,

$$[\psi]^\alpha = \begin{cases} \{t \in \mathbb{R} |\ \psi(t) > \alpha\}, if\ \alpha \in [0,1], \\ cl(supp\ \psi), \quad if\ \alpha = 0, \end{cases} \quad (7)$$

with its closed, bounded interval $[\underline{\psi}(t), \overline{\psi}(t)]$. $\underline{\psi}(t)$, $\overline{\psi}(t)$ are lower and upper bound of $[\psi]^\alpha$ respectively.

Definition 5: Hukuhara differential [27]

A function $f: (a,b) \to R_f$ is called Hukuhara differentiable, if $h > 0$ sufficiently small then H-difference exist $f(t+h) \ominus f(t), f(t) \ominus f(t-h)$ and there exist an element $f'(t) \in R_f$. Such that,

$$\lim_{h\to 0} \frac{f(t+h) \ominus f(t)}{h} = \lim_{h\to 0} \frac{f(t) \ominus f(t-h)}{h} = f'(t)$$

The $f'(t)$ fuzzy number is called Hukuhara derivative of f at t.

III. METHODOLOGY

Given that second-order FODE of the form defined in Equation (1) be a mapping,

$$f : \mathcal{R}_f \to \mathcal{R}_f$$

and $y_0 \in \mathcal{R}_f$ with $\alpha -$ level set

$$(y_0)^\alpha = \left(\left[\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right]\right)^\alpha \quad \alpha \in [0,1]$$

The partition of the $[0,T]$ has the set of grid points

$$0 = t_0 < t_1 < t_2 < t_3 <, \dots, < t_{N-1} < t_N = T$$

with exact solution as

$$(Y(t_n,\alpha))^\alpha = \left([\underline{Y}(t_n,\alpha), \overline{Y}(t_n,\alpha)]\right)^\alpha \quad (8)$$

and approximation solution also denoted as

$$(y(t_n,\alpha))^\alpha = \left([\underline{y}(t_n,\alpha), \overline{y}(t_n,\alpha)]\right)^\alpha \quad (9)$$

at which points, $h = \frac{T-t_0}{N},\ t_n = t_0 + nh,$

$$0 \leq n \leq N$$

The required form of the proposed method in first-order form with second, and third derivative of y is stated below as,

$$(y_{n+\eta})^{\alpha} = \left(y_n + \sum_{i=1}^{3} \left[\sum_{j=0}^{2} \beta_{ij\kappa}\, y_{n+j}^{i} \right] \right)^{\alpha} \tag{10}$$

$$where\ \eta = 1,2$$

By applying Taylor series expansions [28],

$$\big(y(t+h;\alpha)\big)^{\alpha} = \left(\sum_{i=0}^{N} \frac{h^i}{i!} y^i(t;\alpha) \right)^{\alpha} \tag{11}$$

to expand each term in Equation (10) using,

$$(y_{n+j})^{\alpha} = y(t_n + jh;\alpha) = \left(\sum_{i=0}^{n} \frac{(jh)^i}{i!} y^i(t_n;\alpha) \right)^{\alpha}, \qquad j = 0,1,2 \tag{12}$$

$$(y_{n+j})^{\alpha} = \big(y(t_n + jh)\big)^{\alpha} =$$

$$= y(t_n;\alpha) + jhy'(t_n;\alpha) + \frac{(jh)^2}{2!} y''(t_n;\alpha) + \frac{(jh)^3}{3!} y'''(t_n;\alpha) + \cdots$$

where $\beta_{ij\kappa} = A^{-1}D$ with

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & 2h & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & \frac{h^2}{2!} & \frac{(2h)^2}{2!} & 0 & h & 2h & 1 & 1 & 1 \\ 0 & \frac{h^3}{3!} & \frac{(2h)^3}{3!} & 0 & \frac{h^2}{2!} & \frac{(2h)^2}{2!} & 0 & h & 2h \\ 0 & \frac{h^4}{4!} & \frac{(2h)^4}{4!} & 0 & \frac{h^3}{3!} & \frac{(2h)^3}{3!} & 0 & \frac{h^2}{2!} & \frac{(2h)^2}{2!} \\ 0 & \frac{h^5}{5!} & \frac{(2h)^5}{5!} & 0 & \frac{h^4}{4!} & \frac{(2h)^4}{4!} & 0 & \frac{h^3}{3!} & \frac{(2h)^3}{3!} \\ 0 & \frac{h^6}{6!} & \frac{(2h)^6}{6!} & 0 & \frac{h^5}{5!} & \frac{(2h)^5}{5!} & 0 & \frac{h^4}{4!} & \frac{(2h)^4}{4!} \\ 0 & \frac{h^7}{7!} & \frac{(2h)^7}{7!} & 0 & \frac{h^6}{6!} & \frac{(2h)^6}{6!} & 0 & \frac{h^5}{5!} & \frac{(2h)^5}{5!} \\ 0 & \frac{h^8}{8!} & \frac{(2h)^8}{8!} & 0 & \frac{h^7}{7!} & \frac{(2h)^7}{7!} & 0 & \frac{h^6}{6!} & \frac{(2h)^6}{6!} \end{bmatrix},$$

$$D = \begin{bmatrix} \kappa h & \frac{(\kappa h)^2}{2!} & \frac{(\kappa h)^3}{3!} & \frac{(\kappa h)^4}{4!} & \frac{(\kappa h)^5}{5!} & \frac{(\kappa h)^6}{6!} & \frac{(\kappa h)^7}{7!} & \frac{(\kappa h)^8}{8!} & \frac{(\kappa h)^9}{9!} \end{bmatrix}^T$$

Equation (10) takes the form of

$$(y_{n+1})^{\alpha} = (y_n)^{\alpha} + \begin{pmatrix} [\beta_{101}y'_n + \beta_{111}y'_{n+1} + \beta_{121}y'_{n+2}] + \\ [\beta_{201}y''_n + \beta_{211}y''_{n+1} + \beta_{221}y''_{n+2}] + \\ [\beta_{301}y'''_n + \beta_{311}y'''_{n+1} + \beta_{321}y'''_{n+2}] \end{pmatrix}^{\alpha} \tag{13}$$

$$(y_{n+2})^{\alpha} = (y_n)^{\alpha} + \begin{pmatrix} [\beta_{102}y'_n + \beta_{112}y'_{n+1} + \beta_{122}y'_{n+2}] + \\ [\beta_{202}y''_n + \beta_{212}y''_{n+1} + \beta_{222}y''_{n+2}] + \\ [\beta_{302}y'''_n + \beta_{312}y'''_{n+1} + \beta_{322}y'''_{n+2}] \end{pmatrix}^{\alpha}$$

Hence obtaining the coefficients values of $\beta_{ij\kappa}$ as

$$(\beta_{101}, \beta_{111}, \beta_{121}, \beta_{201}, \beta_{211}, \beta_{221}, \beta_{301}, \beta_{311}, \beta_{321})^T = \left(\frac{5669}{13440}, \frac{8192}{13440}, \frac{-421}{13440}, \frac{303}{4480}, \frac{-560}{4480}, \frac{47}{4480}, \frac{169}{40320}, \frac{1024}{40320}, \frac{-41}{40320}\right)^T$$

$$(\beta_{102}, \beta_{112}, \beta_{122}, \beta_{202}, \beta_{212}, \beta_{222}, \beta_{302}, \beta_{312}, \beta_{322})^T = \left(\frac{41}{105}, \frac{128}{105}, \frac{41}{105}, \frac{2}{35}, 0, \frac{-2}{35}, \frac{1}{315}, \frac{16}{315}, \frac{1}{315}\right)^T$$

Substituting in Equation (13) gives the desired two-step block method.

$$(y_{n+1})^{\alpha} = (y_n)^{\alpha} + \Big(\frac{h}{13440}[5669y'_n + 8192y'_{n+1} - 421y'_{n+2}] + \frac{h^2}{4480}[303y''_n - 560y''_{n+1} + 47y''_{n+2}] + \frac{h^3}{40320}[169y'''_n + 1024y'''_{n+1} - 41y'''_{n+2}]\Big)^{\alpha}$$

$$(y_{n+2})^{\alpha} = (y_n)^{\alpha} + \Big(\frac{h}{105}[41y'_n + 128y'_{n+1} + 41y'_{n+2}] + \frac{2h^2}{35}[y''_n - y''_{n+2}] + \frac{h^3}{315}[y'''_n + 16y'''_{n+1} + y'''_{n+2}]\Big)^{\alpha} \tag{14}$$

Then the upper and lower solutions of the proposed method for the approximation solution of FIVPs is obtained from Equation (14) as

$$\left(\underline{y}_{n+1}\right)^\alpha = \left(\underline{y}_n\right)^\alpha + \left(\frac{h}{13440}\left[5669\underline{y}'_n + 8192\underline{y}'_{n+1} - 421\underline{y}'_{n+2}\right]\right)^\alpha + \left(\frac{h^2}{4480}\left[303\underline{y}''_n - 560\underline{y}''_{n+1} + 47\underline{y}''_{n+2}\right]\right)^\alpha + \left(\frac{h^3}{40320}\left[169\underline{y}'''_n + 1024\underline{y}'''_{n+1} - 41\underline{y}'''_{n+2}\right]\right)^\alpha$$

$$\left(\overline{y}_{n+1}\right)^\alpha = \left(\overline{y}_n\right)^\alpha + \left(\frac{h}{13440}\left[5669\overline{y}'_n + 8192\overline{y}'_{n+1} - 421\overline{y}'_{n+2}\right]\right)^\alpha + \left(\frac{h^2}{4480}\left[303\overline{y}''_n - 560\overline{y}''_{n+1} + 47\overline{y}''_{n+2}\right]\right)^\alpha + \left(\frac{h^3}{40320}\left[169\overline{y}'''_n + 1024\overline{y}'''_{n+1} - 41\overline{y}'''_{n+2}\right]\right)^\alpha$$

$$\left(\underline{y}_{n+2}\right)^\alpha = \left(\underline{y}_n\right)^\alpha + \left(\frac{h}{105}\left[41\underline{y}'_n + 128\underline{y}'_{n+1} + 41\underline{y}'_{n+2}\right]\right)^\alpha + \left(\frac{2h^2}{35}\left[\underline{y}''_n - \underline{y}''_{n+2}\right]\right)^\alpha + \left(\frac{h^3}{315}\left[\underline{y}'''_n + 16\underline{y}'''_{n+1} + \underline{y}'''_{n+2}\right]\right)^\alpha$$

$$\left(\overline{y}_{n+2}\right)^\alpha = \left(\overline{y}_n\right)^\alpha + \left(\frac{h}{105}\left[41\overline{y}'_n + 128\overline{y}'_{n+1} + 41\overline{y}'_{n+2}\right]\right)^\alpha + \left(\frac{2h^2}{35}\left[\overline{y}''_n - \overline{y}''_{n+2}\right]\right)^\alpha + \left(\frac{h^3}{315}\left[\overline{y}'''_n + 16\overline{y}'''_{n+1} + \overline{y}'''_{n+2}\right]\right)^\alpha \quad (15)$$

Hence Equation (15) represent the proposed method. The correctors of the block method, takes the form,

$$(A^0Y_{n+k})^\alpha = (A^1Y_{n-k} + h[B^0Y'_{n+k} + B^1Y'_{n-k}] + h^2[C^0Y''_{n+k} + C^1Y''_{n-k}] + h^3[D^0Y'''_{n+k} + D^1Y'''_{n-k}])^\alpha \quad (16)$$

$A^0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $A^1 = \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}$, $B^0 = \begin{pmatrix}\frac{8192}{13440} & \frac{-421}{13440}\\ \frac{128}{105} & \frac{41}{105}\end{pmatrix}$, $B^1 = \begin{pmatrix}0 & \frac{5669}{13440}\\ 0 & \frac{41}{105}\end{pmatrix}$,

$C^0 = \begin{pmatrix}\frac{-560}{4480} & \frac{47}{4480}\\ 0 & \frac{-2}{35}\end{pmatrix}$, $C^1 = \begin{pmatrix}0 & \frac{303}{4480}\\ 0 & \frac{2}{35}\end{pmatrix}$, $D^0 = \begin{pmatrix}\frac{1024}{40320} & \frac{-41}{40320}\\ \frac{16}{315} & \frac{1}{315}\end{pmatrix}$, $D^1 = \begin{pmatrix}0 & \frac{169}{40320}\\ 0 & \frac{1}{315}\end{pmatrix}$,

$(Y_{n+k})^\alpha = \begin{pmatrix}y_{n+1}\\ y_{n+2}\end{pmatrix}^\alpha$, $(Y_{n-k})^\alpha = \begin{pmatrix}y_{n-1}\\ y_n\end{pmatrix}^\alpha$, $(Y'_{n+k})^\alpha = \begin{pmatrix}y'_{n+1}\\ y'_{n+2}\end{pmatrix}^\alpha$, $(Y'_{n-k})^\alpha = \begin{pmatrix}y'_{n-1}\\ y'_n\end{pmatrix}^\alpha$, $(Y''_{n+k})^\alpha = \begin{pmatrix}y''_{n+1}\\ y''_{n+2}\end{pmatrix}^\alpha$, $(Y''_{n-k})^\alpha = \begin{pmatrix}y''_{n-1}\\ y''_n\end{pmatrix}^\alpha$, $(Y'''_{n+k})^\alpha = \begin{pmatrix}y'''_{n+1}\\ y'''_{n+2}\end{pmatrix}^\alpha$, $(Y'''_{n-k})^\alpha = \begin{pmatrix}y'''_{n-1}\\ y'''_n\end{pmatrix}^\alpha$

IV. PROPERTIES OF PROPOSED METHOD

This section will detail the convergence properties of the developed two-step second-third-derivative block method, following the given theorem and definitions.

Theorem 1: [29]

A block method is convergent iff it is consistent and zero-stable

Definition 6:[30]

A block method is consistent if it has order $\rho \geq 1$.

Definition 7: Zero-Stability [30]

Block with matrix difference equation in the following form

$$A^0\hat{Y}_{n+k} = A^1\hat{Y}_{n-k} + B^1\hat{Y}'_{n-k} + B^2\hat{Y}''_{n-k} +, \dots, + B^{(m-1)}\hat{Y}^{(m-1)}_{n-k} + h^m\left[C^0\hat{Y}^m_{n+k} + C^1\hat{Y}^m_{n-k}\right] + h^{(m+1)}\left[D^0\hat{Y}^{(m+1)}_{n+k} + D^1\hat{Y}^{(m+1)}_{n-k}\right] + h^{(m+2)}\left[E^0\hat{Y}^{(m+2)}_{n+k} + E^1\hat{Y}^{(m+2)}_{n-k}\right] \quad (17)$$

Here $\hat{Y}^{(d)}_{n+k} = \left(\hat{y}^{(d)}_{n+1}, \hat{y}^{(d)}_{n+2}, \dots, \hat{y}^{(d)}_{n+k}\right)^T$, and $\hat{Y}^{(d)}_{n-k} = \left(\hat{y}^{(d)}_{n-(k-1)}, \hat{y}^{(d)}_{n-(k-2)}, \dots, \hat{y}^{(d)}_{n}\right)^T$ is zero-stable if the first characteristic polynomial takes form

$$\mathrm{P}(\Psi) = \det\left(\Psi_v A^0 - A^1\right) \quad (18)$$

the root of $\mathrm{P}(\Psi) = 0$ satisfy the $|\Psi_v| \leq 1,\ v = 1, \dots, k$.

These definitions for linear multistep methods in crisp form is adopted to the proposed method for fuzzy initial value problems to prove the convergence properties for the proposed method.

Order and Error Constant

The linear operator associated with Equation (10) is defined as:

$$L([\,y(t),h])^{\alpha} = \left(\,y_n - \sum_{i=1}^{3}\left[\sum_{j=0}^{2}\beta_{ij\kappa}\,y_{n+j}^{i}\right]\right)^{\alpha} \quad (19)$$

$$L[\,y(t;\alpha),h] = B_0\,y(t_n;\alpha) + B_1 h\,y'(t_n;\alpha)$$

$$+B_2 h^2\,y''(t_n;\alpha) + \cdots + B_\rho h^\rho\,y^\rho(t_n;\alpha) + B_{\rho+1}h^{\rho+1}\,y^{\rho+1}(t_n;\alpha)$$

The order of this method is z if $B_0 = B_1 = \cdots = B_z = 0$, $B_{z+1} \neq 0$ and B_{z+1} is constant error.

Expanding y_{n+j}^{i} in Equation (19) by means of Taylor series to get,

$$B_0 = \begin{bmatrix}1-1\\1-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 - \frac{1}{13440}\{5669(1)+8192(1)-421(1)\} \\ 2 - \frac{1}{105}\{41(1)+128(1)+41(1)\} \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$B_2 = \begin{bmatrix} \frac{1}{2!} - \frac{1}{13440}\{8192(1)-421(2)\} - \frac{1}{4480}\{303(1)-560(1)+47(1)\} \\ \frac{2^2}{2!} - \frac{1}{105}\{128(1)+41(2)\} - \frac{2}{35}\{1-1\} \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$B_3 = \begin{bmatrix} \frac{1}{3!} - \frac{1}{13440}\left\{8192\left(\frac{1}{2!}\right) - 421\left(\frac{2^2}{2!}\right)\right\} - \frac{1}{4480}\{560(1)+47(2)\} - \frac{1}{40320}\{169(1)+1024(1)-41(1)\} \\ \frac{2^3}{3!} - \frac{1}{105}\left\{128\left(\frac{1}{2!}\right) + 41\left(\frac{2^2}{2!}\right)\right\} - \frac{2}{35}\{-2\} - \frac{1}{315}\{1+16(1)+1\} \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$B_4 = \begin{bmatrix} \frac{1}{4!} - \frac{1}{13440}\left\{8192\left(\frac{1}{3!}\right) - 421\left(\frac{2^3}{3!}\right)\right\} - \frac{1}{4480}\left\{-560\left(\frac{1}{2!}\right) + 47\left(\frac{2^2}{2!}\right)\right\} - \frac{1}{40320}\{1024(1)-41(2)\} \\ \frac{2^4}{4!} - \frac{1}{105}\left\{128\left(\frac{1}{3!}\right) + 41\left(\frac{2^3}{3!}\right)\right\} - \frac{2}{35}\left\{-\frac{2^2}{2!}\right\} - \frac{1}{315}\{16(1)+2\} \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$B_5 = \begin{bmatrix} \frac{1}{5!} - \frac{1}{13440}\left\{8192\left(\frac{1}{4!}\right) - 421\left(\frac{2^4}{4!}\right)\right\} - \frac{1}{4480}\left\{-560\left(\frac{1}{3!}\right) + 47\left(\frac{2^3}{3!}\right)\right\} - \frac{1}{40320}\{1024\left(\frac{1}{2!}\right) - 41\left(\frac{2^2}{2!}\right)\} \\ \frac{2^5}{5!} - \frac{1}{105}\left\{128\left(\frac{1}{4!}\right) + 41\left(\frac{2^4}{4!}\right)\right\} - \frac{2}{35}\left\{-\frac{2^3}{3!}\right\} - \frac{1}{315}\{16\left(\frac{1}{2!}\right) + \frac{2^2}{2!}\} \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$B_6 = \begin{bmatrix} \frac{1}{6!} - \frac{1}{13440}\left\{8192\left(\frac{1}{5!}\right) - 421\left(\frac{2^5}{5!}\right)\right\} - \frac{1}{4480}\left\{-560\left(\frac{1}{4!}\right) + 47\left(\frac{2^4}{4!}\right)\right\} - \frac{1}{40320}\left\{1024\left(\frac{1}{3!}\right) - 41\left(\frac{2^3}{3!}\right)\right\} \\ \frac{2^6}{6!} - \frac{1}{105}\left\{128\left(\frac{1}{5!}\right) + 41\left(\frac{2^5}{5!}\right)\right\} - \frac{2}{35}\left\{-\frac{2^4}{4!}\right\} - \frac{1}{315}\left\{16\left(\frac{1}{3!}\right) + \frac{2^3}{3!}\right\} \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$B_7 = \begin{bmatrix} \frac{1}{7!} - \frac{1}{13440}\left\{8192\left(\frac{1}{6!}\right) - 421\left(\frac{2^6}{6!}\right)\right\} - \frac{1}{4480}\left\{-560\left(\frac{1}{5!}\right) + 47\left(\frac{2^5}{5!}\right)\right\} - \frac{1}{40320}\left\{1024\left(\frac{1}{4!}\right) - 41\left(\frac{2^4}{4!}\right)\right\} \\ \frac{2^7}{7!} - \frac{1}{105}\left\{128\left(\frac{1}{6!}\right) + 41\left(\frac{2^6}{6!}\right)\right\} - \frac{2}{35}\left\{-\frac{2^5}{5!}\right\} - \frac{1}{315}\left\{16\left(\frac{1}{4!}\right) + \frac{2^4}{4!}\right\} \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$B_8 = \begin{bmatrix} \frac{1}{8!} - \frac{1}{13440}\left\{8192\left(\frac{1}{7!}\right) - 421\left(\frac{2^7}{7!}\right)\right\} - \frac{1}{4480}\left\{-560\left(\frac{1}{6!}\right) + 47\left(\frac{2^6}{6!}\right)\right\} - \frac{1}{40320}\left\{1024\left(\frac{1}{5!}\right) - 41\left(\frac{2^5}{5!}\right)\right\} \\ \frac{2^8}{8!} - \frac{1}{105}\left\{128\left(\frac{1}{7!}\right) + 41\left(\frac{2^7}{7!}\right)\right\} - \frac{2}{35}\left\{-\frac{2^6}{6!}\right\} - \frac{1}{315}\left\{16\left(\frac{1}{5!}\right) + \frac{2^5}{5!}\right\} \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$B_9 = \begin{bmatrix} \frac{1}{9!} - \frac{1}{13440}\left\{8192\left(\frac{1}{8!}\right) - 421\left(\frac{2^8}{8!}\right)\right\} - \frac{1}{4480}\left\{-560\left(\frac{1}{7!}\right) + 47\left(\frac{2^7}{7!}\right)\right\} - \frac{1}{40320}\left\{1024\left(\frac{1}{6!}\right) - 41\left(\frac{2^6}{6!}\right)\right\} \\ \frac{2^9}{9!} - \frac{1}{105}\left\{128\left(\frac{1}{8!}\right) + 41\left(\frac{2^8}{8!}\right)\right\} - \frac{2}{35}\left\{-\frac{2^7}{7!}\right\} - \frac{1}{315}\left\{16\left(\frac{1}{6!}\right) + \frac{2^6}{6!}\right\} \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$B_{10} = \begin{bmatrix} \frac{1}{10!} - \frac{1}{13440}\left\{8192\left(\frac{1}{9!}\right) - 421\left(\frac{2^9}{9!}\right)\right\} - \\ \frac{1}{4480}\left\{-560\left(\frac{1}{8!}\right) + 47\left(\frac{2^8}{8!}\right)\right\} - \\ \frac{1}{40320}\left\{1024\left(\frac{1}{7!}\right) - 41\left(\frac{2^7}{7!}\right)\right\} \\ \frac{2^{10}}{10!} - \frac{1}{105}\left\{128\left(\frac{1}{9!}\right) + 41\left(\frac{2^9}{9!}\right)\right\} - \\ \frac{2}{35}\left\{-\frac{2^8}{8!}\right\} - \frac{1}{315}\left\{16\left(\frac{1}{7!}\right) + \frac{2^7}{7!}\right\} \end{bmatrix} = \begin{bmatrix} 6.9e^{-8} \\ -4.1e^{-20} \end{bmatrix}$$

This proposed method has order z = 9, with error constant $[6.9e^{-8}, -4.1e^{-20}]$.

According to definition 6, hence proposed method is consistent.

Zero-stability

To apply Equation (18) in fuzzy form for zero-stability of the proposed method

$$P(\Psi) = \left[\begin{pmatrix} \Psi & 0 \\ 0 & \Psi^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right]$$

$$P(\Psi) = \Psi(\Psi^2 - 1) = 0$$

$$\Psi = 0, \pm 1$$

The obtained roots are lies on the unit disk so the proposed method is zero-stable.

According to the theorem 1, proposed method is convergent.

Region of Absolute Stability

Obtaining the polynomial for absolute stability region is determined as

$$\left(det\left[-(w)^{\kappa} + A^1 + z\left[\sum_{i=0}^{k} B^i w^{k-i}\right] + z^2\left[\sum_{i=0}^{k} C^i w^{k-i}\right] + z^3\left[\sum_{i=0}^{k} D^i w^{k-i}\right]\right]\right)^{\alpha} \quad (20)$$

$$z = \lambda h$$

So, proposed method polynomial for stability is obtained as:

$$R(w) = \left(\frac{z^6}{7560} - \frac{z^5}{420} + \frac{11z^4}{504} - \frac{99161z^3}{793800} + \frac{577z^2}{1260} - \frac{33011z}{33075} + \frac{314}{315}\right)w^3 - \left(\frac{169z^6}{793800} + \frac{z^5}{304} + \frac{6287z^4}{264600} + \frac{307z^3}{2520} + \frac{11z^2}{24} + z + 1\right)w$$

Roots of the polynomial for stability plotting using locus boundary approach is shown in the figure below.

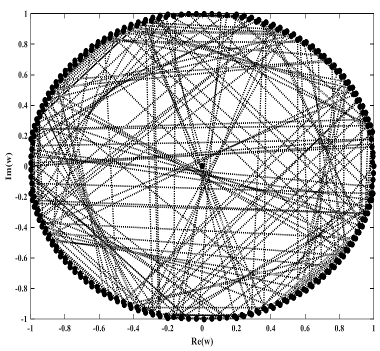

Fig. 1: Absolute stability Region of proposed method

V. RESULTS AND DISCUSSION

This section details the application of the developed Obrechkoff-type two-step implicit block method for the numerical solution of nonlinear FIVPs and the results obtained are compared with the exact solution. Comparison between exact and approximate solutions are shown in tables and graphs.

$x -$ axis show the value of approximation solution

$y -$ axis show the value of $\alpha -$ level set

$\underline{Y}, \overline{Y}$ are exact solution of lower and upper bound respectively

$\underline{y}, \overline{y}$ are approximation solution of lower and upper bound respectively

$\left|\underline{Y} - \underline{y}\right|$ absolute error of lower bound approximation

$\left|\overline{Y} - \overline{y}\right|$ absolute error of upper bound approximation

h is the step size

Example 1. [5]

Consider the non-linear Bernoulli FIVP with triangular and trapezoidal fuzzy numbers

$$y'(t) + ty(t) = 2ty^2(t) \quad t \in [0,1]$$

$$y(0, \alpha) = [3 - \alpha, 1 + \alpha] \; and \left[3 - \frac{3}{2}\alpha, 1 - \frac{1}{2}\alpha\right],$$

with exact solution

$Y(t, \alpha) =$

$$\left[\left(2+(3-\alpha)e^{\frac{t^2}{2}}\right)^{-1},\left(2+(1+\alpha)e^{\frac{t^2}{2}}\right)^{-1}\right] \text{ and}$$

$$\left[\left(2+\left(3-\frac{3}{2}\alpha\right)e^{\frac{t^2}{2}}\right)^{-1},\left(2+\left(1-\frac{1}{2}\alpha\right)e^{\frac{t^2}{2}}\right)^{-1}\right]$$

Which at $t=1$,

$$Y(1,\alpha)=\left[\underline{Y}(1,\alpha),\overline{Y}(1,\alpha)\right] \text{ where } 0<\alpha\leq 1$$

The exact and approximate solutions shown in Tables I, and II, represents the lower and upper solution of the given FIVPs respectively.

Figure 2,3 represents the complete iterations graphs of triangular and trapezoidal number respectively with the step-size $h=0.1$ partition of the time interval $t\in[0,1]$.

TABLE I
Lower/Upper Solution with Triangular Fuzzy Number of Example 1

α	$\underline{y}$	$\lvert\underline{Y}-\underline{y}\rvert$
0	0.143964356017342450	5.551115e-17
0.1	0.147464531308613340	8.326673e-17
0.2	0.151139145763040220	1.110223e-16
0.3	0.155001572988822610	1.665335e-16
0.4	0.159066589520268980	1.942890e-16
0.5	0.163350563735510070	2.220446e-16
0.6	0.167871676146674360	2.498002e-16
0.7	0.172650177313688980	3.053113e-16
0.8	0.177708691098104750	3.608225e-16
0.9	0.183072572836609160	3.608225e-16
1	0.188770334399073120	4.163336e-16
α	$\overline{y}$	$\lvert\overline{Y}-\overline{y}\rvert$
0	0.274068619061197450	4.996004e-16
0.1	0.262219879178707260	5.551115e-16
0.2	0.251353189755283870	5.551115e-16
0.3	0.241351317225855620	5.828671e-16
0.4	0.232114973806293650	5.828671e-16
0.5	0.223559510199762830	5.551115e-16
0.6	0.215612313718264230	5.828671e-16
0.7	0.208210742310189520	5.273559e-16
0.8	0.201300469991207790	4.718448e-16
0.9	0.194834151170723640	4.440892e-16
1	0.188770334399073120	4.163336e-16

TABLE II
Lower/Upper Solution with Trapezoidal Fuzzy Number of Example 1

α	$\underline{y}$	$\lvert\underline{Y}-\underline{y}\rvert$
0	0.143964356017342450	5.551115e-17
0.1	0.149279228647828920	1.110223e-16
0.2	0.155001572988822610	1.665335e-16
0.3	0.161180115928929310	2.220446e-16
0.4	0.167871676146674360	2.498002e-16
0.5	0.175142916560684790	3.053113e-16
0.6	0.183072572836609160	3.608225e-16
0.7	0.191754315981089580	4.163336e-16
0.8	0.201300469991207790	4.718448e-16
0.9	0.211846898142716120	5.551115e-16
1	0.223559510199762830	5.551115e-16
α	$\overline{y}$	$\lvert\overline{Y}-\overline{y}\rvert$
0	0.274068619061197450	4.996004e-16
0.1	0.280403821318293330	4.440892e-16
0.2	0.287038835142562290	3.885781e-16
0.3	0.293995459079036570	3.885781e-16
0.4	0.301297657383730270	3.330669e-16
0.5	0.308971835831501340	2.775558e-16
0.6	0.317047160775197400	2.220446e-16
0.7	0.325555929581395400	1.665335e-16
0.8	0.334534002360660050	1.110223e-16
0.9	0.344021307160407800	1.110223e-16
1	0.354062433629710850	0.000000e+00

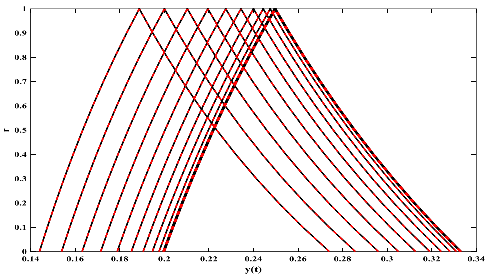

Fig. 2 Example 1 with triangular fuzzy number

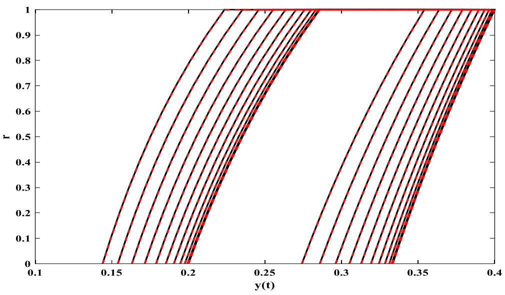

Fig. 3 Example 1 with trapezoidal fuzzy number

Example 2. [31]

Consider the non-linear FIVP with triangular and trapezoidal fuzzy numbers

$$y'(t)=y^2(t)+1$$

$y(0,\alpha)=$

$\left[\frac{1}{10}\alpha-\frac{1}{10},\frac{1}{10}-\frac{1}{10}\alpha\right]$ and $\left[\frac{1}{100}\alpha-\frac{1}{10},\frac{1}{10}-\frac{1}{100}\alpha\right]$

with exact solution

$Y(t,\alpha)=$

$\left[Tan\left(t-arcTan\left(\frac{1-\alpha}{10}\right)\right),Tan\left(t+arcTan\left(\frac{1-\alpha}{10}\right)\right)\right]$ and

$\left[Tan\left(t-arcTan\left(\frac{(10-\alpha)}{100}\right)\right),Tan\left(t+arcTan\left(\frac{(10-\alpha)}{100}\right)\right)\right]$

Which at $t=1$,

$Y(1,\alpha)=[\underline{Y}(1,\alpha),\overline{Y}(1,\alpha)]$ where $0<\alpha\le 1$

The exact and approximate solutions shown in Tables III, and IV represents the lower and upper solution of the given FIVPs respectively.

Figure 4,5 represents the complete iterations graphs of triangular and trapezoidal number respectively with the step-size $h=0.1$ partition of the time interval $t\in[0,1]$.

TABLE III
Lower/Upper Solution with Triangular Fuzzy Number of Example 2

α	$\underline{y}$	$\lfloor \underline{Y}-\underline{y} \rfloor$
0	1.261016102725076800	8.881784e-16
0.1	1.287011566649140200	1.332268e-15
0.2	1.313727034166937100	1.332268e-15
0.3	1.341192838562831100	1.110223e-15
0.4	1.369441041285456700	1.332268e-15
0.5	1.398505556798372600	1.554312e-15
0.6	1.428422288413147400	1.554312e-15
0.7	1.459229276248731600	1.332268e-15
0.8	1.490966858599166200	1.554312e-15
0.9	1.523677848148974600	1.554312e-15
1	1.557407724654902300	1.332268e-15
α	$\overline{y}$	$\lceil \overline{Y}-\overline{y} \rceil$
0	1.963150263095175200	2.442491e-15
0.1	1.915961751533205900	1.776357e-15
0.2	1.870452269733388100	1.998401e-15
0.3	1.826533770950384600	1.998401e-15
0.4	1.784124258798744300	1.776357e-15
0.5	1.743147276322709200	1.776357e-15
0.6	1.703531445981465000	1.998401e-15
0.7	1.665210054727049900	1.554312e-15
0.8	1.628120679100946200	1.554312e-15
0.9	1.592204845917552000	1.554312e-15
1	1.557407724654902300	1.332268e-15

TABLE IV
Lower/Upper Solution with Trapezoidal Fuzzy Number of Example 2

α	$\underline{y}$	$\lfloor \underline{Y}-\underline{y} \rfloor$
0	1.261016102725076800	8.881784e-16
0.1	1.263584079625319000	1.110223e-15
0.2	1.266158996135131600	8.881784e-16
0.3	1.268740880422606400	1.110223e-15
0.4	1.271329760808488600	1.110223e-15
0.5	1.273925665767212700s	1.110223e-15
0.6	1.276528623927945600	1.110223e-15
0.7	1.279138664075639900	1.110223e-15
0.8	1.281755815152094700	8.881784e-16
0.9	1.284380106257024800	1.110223e-15
1	1.287011566649140200	1.332268e-15
α	$\overline{y}$	$\lceil \overline{Y}-\overline{y} \rceil$
0	1.963150263095175200	2.442491e-15
0.1	1.958353212274391200	1.554312e-15
0.2	1.953573794667807600	2.220446e-15
0.3	1.948811913228351700	1.998401e-15
0.4	1.944067471619798100	1.998401e-15
0.5	1.939340374210271700	1.998401e-15
0.6	1.934630526065822500	2.442491e-15
0.7	1.929937832944070200	1.776357e-15
0.8	1.925262201287917500	1.554312e-15
0.9	1.920603538219334000	2.220446e-15
1	1.915961751533205900	1.776357e-15

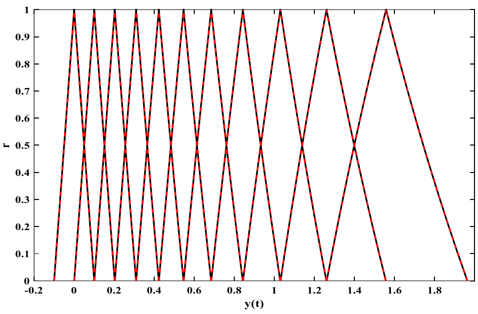

Fig. 4 Example 2 with triangular fuzzy number

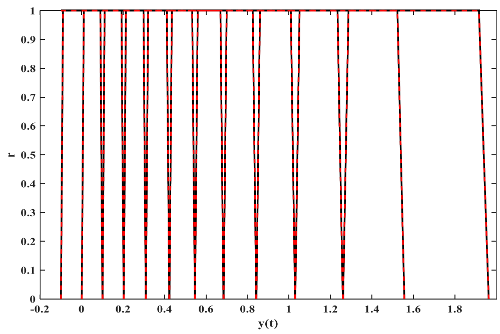

Fig. 5 Example 2 with trapezoidal fuzzy number

Example 3. [32]

Consider the crisp non-linear IVP

$$y'(t)=-y(t)-y^2(t)\; y(0)=1$$

According to Jameel [33], the crisp equation can be model in a fuzzy version by using the definition of the fuzzy theory. In this example according to [33], difuzzify the initial condition $[1]_r$ as both triangular and trapezoidal fuzzy numbers. So, ca writes the crisp initial condition in fuzzy form follow as

$y(0,\alpha)=\left[\frac{1}{\alpha-4},\frac{1}{-2\alpha-1}\right]$ and $\left[\frac{2}{\alpha-8},\frac{1}{-1.5\alpha-1}\right]$

with exact solution

$Y(t,\alpha)=\left[\left(\frac{e^{-t}}{\alpha-3-e^{-t}}\right),\left(\frac{e^{-t}}{-2\alpha-e^{-t}}\right)\right]$and $\left[\left(\frac{e^{-t}}{0.5\alpha-3-e^{-t}}\right),\left(\frac{e^{-t}}{-1.5\alpha-e^{-t}}\right)\right]$

Which at $t=1$,

$$Y(1,\alpha) = [\underline{Y}(1,\alpha), \overline{Y}(1,\alpha)] \text{ where } 0 < \alpha \le 1$$

The exact and approximate solutions shown in the Tables V, and VI represents the lower and upper solution of the given FIVPs respectively.

Figure 6,7 represents the complete iterations graphs of triangular and trapezoidal number respectively with the step-size $h = 0.05$ partition of the time interval $t \in [0,1]$.

TABLE V
Lower/Upper Solution with Triangular Fuzzy Number of Example 3

α	$\underline{y}$	$\left\lvert\underline{Y}-\underline{y}\right\rvert$
0	-0.10923177257303593	0.000000e+00
0.1	-0.11257436138450932	0.000000e+00
0.2	-0.11612798024769690	0.000000e+00
0.3	-0.11991326524583731	0.000000e+00
0.4	-0.12395363371843628	0.000000e+0
0.5	-0.12827576915896249	2.775558e-17
0.6	-0.13291021122500396	0.000000e+00
0.7	-0.13789207844036225	0.000000e+00
0.8	-0.14326195976070402	0.000000e+00
0.9	-0.14906702289995938	0.000000e+00
1	-0.15536240349696362	0.000000e+00
α	$\overline{y}$	$\lceil\overline{Y}-\overline{y}\rceil$
0	-1.00000000000000000	0.000000e+00
0.1	-0.64781257164824857	0.000000e+00
0.2	-0.47908489464208343	5.551115e-17
0.3	-0.38008808279488926	0.000000e+00
0.4	-0.31499778847244764	0.000000e+00
0.5	-0.26894142136999510	0.000000e+00
0.6	-0.23463503092851382	0.000000e+00
0.7	-0.20809079658037991	2.775558e-17
0.8	-0.18694206234120239	0.000000e+00
0.9	-0.16969552558358772	2.775558e-17
1	-0.15536240349696362	0.000000e+00

TABLE VI
Lower/Upper Solution with Trapezoidal Fuzzy Number of Example 1

α	$\underline{y}$	$\left\lvert\underline{Y}-\underline{y}\right\rvert$
0	-0.1092317725730359	0.00000e+00
0.1	-0.1108778807953176	0.00000e+00
0.2	-0.1125743613845093	0.00000e+00
0.3	-0.1143235624257939	0.00000e+00
0.4	-0.1161279802476969	0.00000e+00
0.5	-0.1179902713086377	0.00000e+00
0.6	-0.1199132652458373	0.00000e+00
0.7	-0.1218999792213845	0.00000e+00
0.8	-0.1239536337184362	0.00000e+00
0.9	-0.1260776699614942	2.77558e-17
1	-0.1282757691589624	2.75558e-17
α	$\overline{y}$	$\lceil\overline{Y}-\overline{y}\rceil$
0	-1.00000000000000000	0.000000e+00
0.1	-0.71035729925733238	0.000000e+00
0.2	-0.55081713628763873	0.000000e+00
0.3	-0.44979666030549864	5.551115e-17
0.4	-0.38008808279488926	0.000000e+00
0.5	-0.32908686538320808	0.000000e+00
0.6	-0.29015332942976380	0.000000e+00
0.7	-0.25945749017102809	0.000000e+00
0.8	-0.23463503092851382	0.000000e+00
0.9	-0.21414741474557783	5.551115e-17
1	-0.19695031331397192	2.775558e-17

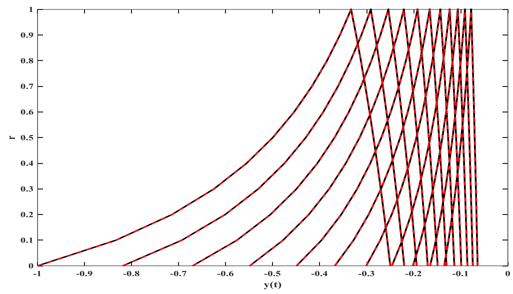

Fig. 6 Example 3 with triangular fuzzy number

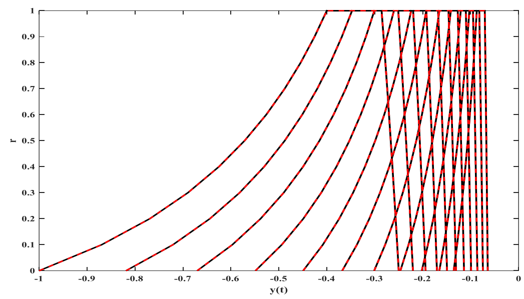

Fig. 7 Example 3 with trapezoidal fuzzy number

Example 4. [34]

Consider the non-linear FIVP with triangular and trapezoidal fuzzy numbers

$$y'(t) = ty^2(t)$$

$$y(0,\alpha) = \left[\frac{2}{\alpha+1}, \frac{2}{3-r}\right] \text{ and } \left[\frac{4}{\alpha+2}, \frac{4}{6-r}\right]$$

with exact solution

$$Y(t,\alpha) = \left[\left(\frac{2}{\alpha+1-t^2}\right), \left(\frac{2}{3-\alpha-t^2}\right)\right] \text{ and } \left[\left(\frac{2}{0.5\alpha+1-t^2}\right), \left(\frac{2}{3-0.5\alpha-t^2}\right)\right]$$

Which at $t = 0.5$,

$$Y(0.5,\alpha) = [\underline{Y}(0.5,\alpha), \overline{Y}(0.5,\alpha)]$$

The exact and approximate solutions shown in Tables VII and VIII represents the lower and upper solution of the given FIVPs respectively.

Figure 8,9 represents the complete iterations graphs of triangular and trapezoidal number respectively with the step-size $h = 0.05$ partition of the time interval $t \in [0,0.5]$.

TABLE VII
Lower/Upper Solution with Triangular Fuzzy Number of Example 4

α	$\underline{y}$	$\lvert \underline{Y} - \underline{y} \rvert$
0	2.666666666666666100	4.440892e-16
0.1	2.352941176470587900	0.000000e+00
0.2	2.105263157894736700	0.000000e+00
0.3	1.904761904761904700	0.000000e+00
0.4	1.739130434782608600	2.220446e-16
0.5	1.600000000000000100	0.000000e+00
0.6	1.481481481481481400	0.000000e+00
0.7	1.379310344827586300	0.000000e+00
0.8	1.290322580645161300	0.000000e+00
0.9	1.212121212121212200	0.000000e+00
1	1.142857142857142800	0.000000e+00
α	$\overline{y}$	$\lceil \overline{Y} - \overline{y} \rceil$
0	0.727272727272727290	0.000000e+00
0.1	0.754716981132075530	0.000000e+00
0.2	0.784313725490196070	1.110223e-16
0.3	0.816326530612244920	1.110223e-16
0.4	0.851063829787234050	0.000000e+00
0.5	0.888888888888888840	0.000000e+00
0.6	0.930232558139534870	1.110223e-16
0.7	0.975609756097560950	0.000000e+00
0.8	1.025641025641025500	0.000000e+00
0.9	1.081081081081081100	2.220446e-16
1	1.142857142857142800	0.000000e+00

TABLE VIII
Lower/Upper Solution with Trapezoidal Fuzzy Number of Example 4

α	$\underline{y}$	$\lvert \underline{Y} - \underline{y} \rvert$
0	2.666666666666666100	4.440892e-16
0.1	2.499999999999999600	4.440892e-16
0.2	2.352941176470587900	0.000000e+00
0.3	2.222222222222222300	0.000000e+00
0.4	2.105263157894736700	0.000000e+00
0.5	2.000000000000000000	0.000000e+00
0.6	1.904761904761904700	0.000000e+00
0.7	1.818181818181818100	0.000000e+00
0.8	1.739130434782608600	2.220446e-16
0.9	1.666666666666666700	0.000000e+00
1	1.600000000000000100	0.000000e+00
α	$\overline{y}$	$\lceil \overline{Y} - \overline{y} \rceil$
0	0.727272727272727290	0.000000e+00
0.1	0.740740740740740700	0.000000e+00
0.2	0.754716981132075530	0.000000e+00
0.3	0.769230769230769270	1.110223e-16
0.4	0.784313725490196070	1.110223e-16
0.5	0.800000000000000040	0.000000e+00
0.6	0.816326530612244920	1.110223e-16
0.7	0.833333333333333370	0.000000e+00
0.8	0.851063829787234050	0.000000e+00
0.9	0.869565217391304320	1.110223e-16
1	0.888888888888888840	0.000000e+00

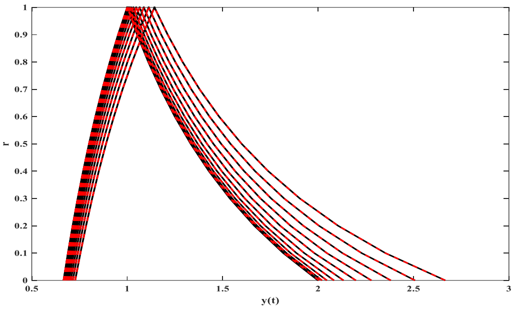

Fig. 8 Example 4 with trapezoidal fuzzy number

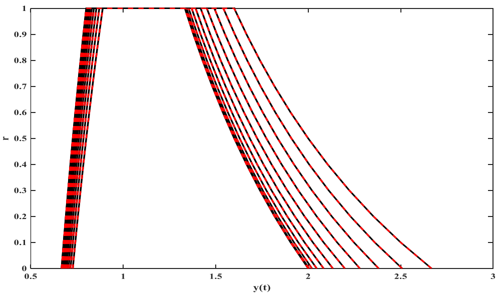

Fig. 9 Example 4 with trapezoidal fuzzy number

In existing literature, most researchers considered the numerical solution of linear fuzzy differential equations rather than the nonlinear form. In this article, a new numerical method is developed to solve nonlinear fuzzy differential equations as shown with examples above. In Example 1, presented the nonlinear fuzzy differential equation [5], although the numerical solution was not considered. Hence, Example 1 is solved approximately using the developed method in this article and compared with the exact solution. From Tables (1-4) and the Figures (2-5), it is seen that the accuracy of the obtained results in term of absolute error is impressive.

For Example, 2, implemented the solution using Runge Kutta method with six-stages and order five [31]. The authors computed the approximate solution with very small step-size ($h = 0.001$) and their obtained results had accuracy in term of absolute error to 10^{-15}. Although, the accuracy seems good, the time taken to implement the required iterations using $h = 0.001$ involves a lot of computational burden. For this reason, Example 2 is solved using the developed method with a larger step-size ($h = 0.01$) compared to [31], and the obtained results have accuracy in term of absolute error to 10^{-16}. The new block method achieves good accuracy with fewer iterations which saves time with low computing burden. The results and graphs are presented in Tables (5-8) and Figures (6-9).

Example 3 was considered by Al-Omari [32] as a crisp first-order nonlinear initial value problem. In this example, the definition of fuzzy numbers in Section 2 is adopted to difuzzify the initial condition as fuzzy numbers, hence having the form of a nonlinear FIVP. Thereafter, Example 3 is solved using the developed method and the obtained results are compared with the exact solution as shown in Tables (9-12) and Figures (10-13), which indicates that the accuracy of the proposed method in terms of absolute error.

Example 4 was solved using the artificial neural network approach [34], and the obtained results have accuracy in term of absolute error to 10^{-5}. In this existing study author obtained the results in a small interval [0 0.2] and compared with the exact solution which not shown article. This implies that that artificial neural network approach limited to specific points. In this article, Example 4 is written with exact solution provided and solved by the proposed method in the interval $[0,0.5]$ with greater accuracy. The obtained results of Example 4 are presented in Tables (13-16), Figures (14-17).

In addition, each example in this article is solved by proposed method using both triangular and trapezoidal fuzzy numbers in the given interval. Figures (2-13) displays the exact and approximate solutions with both triangular and trapezoidal fuzzy numbers. Based on the numerical solutions, the numerical results provided by the developed method in this article is very accurate. From the tables, it has been found that the error range is small with step size $h = 0.1$, and as $h \to 0$ then the error is much smaller. For this reason, Obrechkoff-type two-step implicit block method with the presence of second and third derivative method is an effective iterative method for the numerical solution of first order nonlinear FIVPs.

VI. CONCLUSION

The main goal of this article is to develop a numerical method for the solution of first-order nonlinear FIVPs with the aim of obtaining better accuracy of the solution in terms of absolute error when compared to existing studies. Therefore, for the solution of nonlinear fuzzy ordinary differential equation with fuzzy initial value problem, block method with higher derivatives have proven to be an effective approach with improved accuracy. One of the benefits of the Obrechkoff-type two-step implicit block method as an effective method is its high accuracy with a smaller area of predicted uncertainty in solutions, while also being self-starting. The method was developed by using linear block approach with low computational complexity, and likewise satisfied the convergence conditions for the linear multistep methods. The solution of the nonlinear FODEs with a FIVPS, as seen in the tables, and graphs, demonstrates the applicability of the Obrechkoff-type two-step implicit block method for first-order nonlinear differential equation with fuzzy initial value problem. As a result, this proposed method is a suitable method for first-order nonlinear FIVPs.

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