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A Numerical Analysis of Comparing Several Practical Approaches for Pricing both American and European Put Options

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***Abstract*— Numerical methods are essential in option pricing for handling complex derivatives, multi-asset scenarios, and American options, where analytical solutions are impractical. They offer flexibility, accuracy, and the ability to model real-world market conditions. The present paper compares numerical pricing methods for European and American put option that have yet to be studied. This study implements five basic computational methods. Computer Algebra System (CAS) Python is utilized for the simulations. For both European and American put options, a tabular and graphical analysis of various strategies is offered. The findings show that the Crank-Nicolson Finite Difference Method (CNFDM) yields better results than the other approaches.**

***Index Terms*— American option, Binomial method, Black-Scholes-Merton method, European option, Finite Difference method, Monte Carlo method, Trinomial method.**

I. INTRODUCTION

In modern finance, derivatives like options are traded on many exchanges worldwide. Since the pricing option is a challenging task, it attracts the attention of many researchers nowadays. In most circumstances, many prices must be calculated rapidly; therefore, precise option price computation is essential. At the beginning of the 1970s, Fischer Black, Myron Scholes, and Robert Merton established the Black-Scholes-Merton (BSM) model [1]. One may immediately compute the option value with this method, which is why it is widely used. Jodar et al. [2] explain this model utilizing Mellin transformation. Gonzalez et al. [3] studied the Black-Scholes-Merton model, modified using a distinct dividend. Karagozodu et al. [4] studied the evolution of option valuation models from Black-Scholes-Merton to contemporary approaches, covering various assets and dynamic conditions. Morales [5] developed a modified BSM model inspired by conformable calculus, offering greater flexibility for pricing derivatives in well-developed markets. Parameswaran et al. [6] examine the effects of the BSM option valuation model on call-and-pull option Greeks and the likelihood that they will be executed at expiry. The Monte Carlo (MC) model is a numerical technique employed to value financial derivatives, particularly options, via approximating the underlying asset values' unpredictable trajectories. Phelim Boyle first introduced the Monte Carlo model in 1977. Mahboubeh and Mahnaz [7] studied an adaptive Monte Carlo algorithm for European and American options. This MC [8-12] method is beneficial for valuing complex derivatives with impractical analytical solutions. An approach to value option was developed in 1979 is known binomial model. Mou et al. [13] searched the American put option pricing mechanism under binomial consideration. Xiaoping and Jie [14] investigated the binomial tree and valuing of American-style options. Leduc and Merima [15] found that Joshi's first split tree performed far better for American options due to an optimized form of the tree. Muroi and Suda [16] discussed a new European option pricing method using a binomial tree model and discrete cosine transform. Phelim Boyle first used the trinomial model for option pricing in 1986, which was deemed greater in efficiency than the binomial model and would calculate similar results in fewer steps. After that, Han established an efficient method of quantitatively pricing options and detailed the findings for a continuous distribution of underlying stock price fluctuations known as the trinomial method. The trinomial option valuation model is for options that involve three alternative values for an underlying asset in a single period. Xiaoping et al. [17] investigated pricing options based on the trinomial Markov tree. European put option non-arbitrage valuing and hedging techniques using the trinomial model are suggested by Lin et al. [18]. Leduc [19] studied the trinomial tree using Boyle–Romberg method, which is really helpful for pricing double barrier options. Josheski et al. [20] compare binomial and trinomial models

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with the BSM model for valuing option and their convergence. Hossein et al. [21] researched building trinomial models on Wiener space using the cubature method: applications to financial derivative pricing. The finite difference method [22] for option valuation discretizes partial differential equations and solves them numerically on a grid. Schwartz [23] initially applied the finite difference technique (FDM) to option pricing. Merton [24] was the first to suggest a closed-form solution. Fadugba and Nwozo [25] demonstrated using CNFDM for option valuation. They demonstrate CNFDM's correctness, uniformity, and steadiness in the value of European options. Kumar et al. [26] used BANKNIFTY to compare algebraic BSM equations and computational solutions for driving option pricing. Courtadon [27] presented a framework that relied on traditional numerical methods and a finite difference approach to the option valuation challenge. Nwobi et al. [28] have inspected the effect of CNFDM on option values. They attempted to pinpoint the origins of underpricing in their article. Anwar and Abdallah [29] examined the consistency and option valuation technique using different FDM schemes. Sunday et al. [30] investigated the Finite Difference and Monte Carlo methods for estimating the European Option. When pricing the European option, they found that the Crank-Nicolson technique is stable and corresponds quickly to the Monte Carlo method. Mohammad et al. [31] proposed a qualitatively stable nonstandard Finite Difference Scheme to evaluate the nonlinear Black-Scholes Equation. Mohammad et al. [32] looked at a modified BSM with a different finite difference method.

The literature survey shows many studies have been done on option pricing using different methods. Most of the works analyze European and American call options. There is some scope for working with put options with different methods. So, in this paper, five methods are discussed and compared for the American and the European put options with numerical and graphical descriptions.

II. Option Pricing Models

One needs to formulate an exact model to understand a problem's situation, which is tricky. That is why we use a simplified model that approximates the natural phenomenon ideally. This study describes various mathematical option pricing approaches, specifically:

- Black-Scholes Merton Model (BSM).
- Monte Carlo Method (MC).
- Binomial Option Pricing Model (BOPM).
- Trinomial Option Pricing Model (TOPM).
- Finite Difference Method (FDM).

A. Black-Scholes-Merton Model

The trader's value and hedge derivatives have experienced significant changes after the creation of the BSM model. If the value of an asset is S and an option value is V, then

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \quad (1)$$

Here, The strike price is E , r signifies the risk-free rate of interest, and σ stands for the stock's volatility.

Equation (1) contains a substantial amount of solutions. To obtain particular derivatives equation (1) may be solved. Call options payoff is:

$$C(S,T)=\max(S(T)-E,0) \quad (2)$$

Evident that $S(t) = 0 \; \forall \, t \text{ if } S = 0$; thus, the payment at maturity is zero. So,

$$C(0,t)=0 \text{, for all } 0\le t\le T \text{ .} \quad (3)$$

Conversely, in case the stock's price ever becomes exceedingly high, it will most likely stay extremely high and overwhelm the exercise price, causing it to collapse. So,

$$C(S,T) \approx S \text{, for large } S. \quad (4)$$

Equation (2) is the final condition and other constraints, (3) and (4), are known as boundary conditions. Now the put payoff is:

$$P(S,T)=\max(E-S(T),0) \quad (5)$$

The payoff will be E at time T; when the asset's value is 0. To obtain $P(0,t)$, and we make up for depreciation to obtain

$$P(0,t)=Ee^{-r(T-t)} \text{, for all } 0\le t\le T \quad (6)$$

For extensive S, The payout is almost certainly zero. so

$$P(S,T) \approx 0 \text{, for large } S. \quad (7)$$

Black-Scholes option pricing Formula

Imposing (2), (3), and (4) equation (1) and find a special solution for the value of the call option, which is,

$$C(S,t)=SN(d_1)-Ee^{-r(T-t)}N(d_2)$$

Now from the general put-call parity relation, we get

$$P(S,t)=Ee^{-r(T-t)}N(-d_2)-SN(d_1)$$

where, $N(.)$ is the Normal distribution with mean 0 and variance 1 and

$$d_1=\frac{\log\left(\frac{S}{E}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2=\frac{\log\left(\frac{S}{E}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}=d_1-\sigma\sqrt{T}$$

In this case, C represents the call option price, P the put option price, S the stock price at the moment, t the option maturity period, E the strike price, and r the risk-free interest rate.

B. Monte Carlo Method

A useful numerical technique while using a closed-form approach is not accessible is the Monte Carlo technique. In a world without risk, the expected return is computed with a sampling technique. The main steps are [30]:

- For the chosen period, simulate the movement of the underlying asset in a risk-neutral situation.
- Discount the reward according to the route at the interest rate that is risk-free.
- Carry out the method on a big number of example routes.
- To find the value of the option, across the tested paths, average the cash flow discounts.

The geometric Brownian motion is used in the Monte Carlo analysis of the stock price.

$$dS=\mu Sdt+\sigma Sdw(t) \quad (8)$$

where S is the stock price, $dw(t)$ is a Brownian motion. When ΔS represent the rise in the stock price within the next brief period of Δt, then

$$\frac{\Delta S}{S} = \mu S dt + \sigma z\sqrt{\Delta t} \quad (9)$$

If σ is volatility, μ is the projected return, and z has a normal distribution with mean zero and variance one then equation (9) became

$$S(t+\Delta t) - S(t) = \mu S(t)\Delta t + \sigma S(t) z\sqrt{\Delta t} \quad (10)$$

Instead of using S It is more precise to work with $\ln S$ and utilizing Ito's lemma, we alter the asset pricing process.

$$d(\ln S) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dw(t)$$

From (10),

$$\ln S(t+\Delta t) - \ln S(t) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma z\sqrt{\Delta t}$$

Or

$$S(t+\Delta t) = S(t)e^{\left[\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma z\sqrt{\Delta t}\right]}$$

(11)

This tactic works well when the financial derivative's payout is contingent on the trajectory of the underlying asset over the option's duration [33]. At the maturity date, the price is:

$$S_T^j = Se^{\left[\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma z\sqrt{\Delta t}\right]} \quad (12)$$

where $j = 1, 2, \ldots\ldots, M$ and M is the total amount of attempts. The calculated European call and put option value is

$$C = \frac{1}{M}\sum_{j=1}^{M} e^{-rT}\max[S_T^j - S_t, 0]$$

where S_t is the striking price, which may be found using the geometric or arithmetic means.

C. Binomial Options Pricing Model (BOPM)

In binomial process, two things may happen: either a move up S_u or a move down S_d, where $u > 1$ and $d < 1$ [34]. The shift from S to S_u is known as the up shift, and the shift from S_d is known as the down shift. The chance of an upward shift is P, while the chances of a downward shift is $(1-P)$. BOPM divides the whole-time frame into small intervals of size δt. The elements P, u, and d must produce the right mean and variance values for the period δt. When there is no dividend q from the underlying asset, the expected return will be $(r-q)$. The expected value after a specific time of size δt must be $Se^{(r-q)\delta t}$ [35]. So,

$$Se^{(r-q)\delta t} = PSu + (1-P)Sd$$

$$\Rightarrow e^{(r-q)\delta t} = Pu + (1-P)d \quad (13)$$

For the variable x variance is $E(x^2) - [E(x)]^2$. When the asset price movements as a proportion is R at time δt, and there is a chance P that $(1+R)$ is u and a probability $(1-P)$ is d, the variance of $(1+R)$ is $Pu^2 + (1-P)d^2 - e^{2(r-q)\delta t}$. The variance of $(1+R)$ is equal to the variance of R, which is $\sigma^2\delta t$.

$$Pu^2 + (1-P)d^2 - e^{2(r-q)\delta t} = \sigma^2\delta t$$

(14)

From equation (13)

$$e^{(r-q)\delta t}(u+d) = Pu^2 + (1-P)d^2 + ud$$

so,

$$e^{(r-q)\delta t}(u+d) - ud - e^{2(r-q)\delta t} = \sigma^2\delta t \quad (15)$$

Considering $ud = 1$ [36], hence the relationship can be found by computing the above formula [37]

$$P = \frac{a-d}{u-d}, u = e^{\sigma\sqrt{\delta t}}, d = e^{-\sigma\sqrt{\delta t}}, a = e^{(r-q)\delta t}$$

Then divide the option's life into n subinterval of length δt. We will mention the j th node at time $i\delta t$ as the (i,j) node (where $0 \le i \le n$ and $0 \le j \le i$). Define $f_{i,j}$ as the option value at the (i,j) node. The asset value at the (i,j) node is $S_0 u^i d^{i-j}$. The payoff from the financial commodity at maturity period T a call option is $\max(S_T - K, 0)$. So

$$f_{n,j} = max(S_0 u^i d^{n-j} - K, 0); j = 0, 1, \ldots\ldots, n$$

Likewise, the payment of a put option at time T is $\max(K - S_T, 0)$ so,

$$f_{n,j} = max(K - S_0 u^i d^{n-j}, 0); j = 0, 1, \ldots\ldots, n$$

Traveling from a node (i,j) to the $(i+1, j+1)$ node at a time $(i+j)\delta t$has a probability of P at time $i\delta t$. Similarly, moving from a node (i,j) to the $(i+1, j)$ node at a time $(i+1)\delta t$ has a chance of $(1-P)$ at time $i\delta t$. Risk-neutral valuation when early exercise is not allowed is,

$$f_{i,j} = e^{-r\delta t}[Pf_{i+1,j+1} + (1-P)f_{i+1,j}] \quad \text{For} \quad 0 \le i \le n-1$$

and$0 \le j \le i$.

However, when assessing premature execution, like the American option, we ought to use $f_{i,j}$, thus for a call option:

$$f_{i,j} = \max\{S_0 u^i d^{i-j} - K, e^{-r\delta t}[Pf_{i+1,j+1} + (1-P)f_{i+1,j}]\}$$

And for put option:

$$f_{i,j} = \max\{K - S_0 u^i d^{i-j}, e^{-r\delta t}[Pf_{i+1,j+1} + (1-P)f_{i+1,j}]\}$$

D. Trinomial Option Pricing Model (TOPM)

The TOPM is a financial theory that considers three alternative values for an asset class in a one-time frame: S_u, S_d, and S. The probabilities of these movements are P_u, P_m, and P_d. After the first interval, the projected asset values in the risk-neutral state is $S_0 e^{r\delta t}$ where $\left(\delta t = \frac{T}{N}\right)$; that is,

$$S_0 e^{r\delta t} = P_u S_0 u + P_m S_0 m + P_d S_0 d$$

$$\Rightarrow e^{r\delta t} = P_u u + P_m m + P_d d \quad (16)$$

Further, two requirements are derived from the dispersion and frequency condition;

$$P_u u^2 + P_m m^2 + P_d d^2 - \left(e^{r\delta t}\right)^2 = \sigma^2\delta t \quad (17)$$

$$P_u + P_m + P_d = 1 \quad (18)$$

The following relation can be found after solving equations (16), (17), and (18);

$$P_u = \left(\frac{e^{\frac{r\delta t}{2}} - e^{-\sigma\sqrt{\frac{\delta t}{2}}}}{e^{\sigma\sqrt{\frac{\delta t}{2}}} - e^{-\sigma\sqrt{\frac{\delta t}{2}}}}\right); P_d = \left(\frac{e^{\sigma\sqrt{\frac{\delta t}{2}}} - e^{\frac{r\delta t}{2}}}{e^{\sigma\sqrt{\frac{\delta t}{2}}} - e^{-\sigma\sqrt{\frac{\delta t}{2}}}}\right); P_m = 1 - P_u - P_d;$$

where, $u = e^{\sigma\sqrt{2\delta t}}$, $d = e^{-\sigma\sqrt{2\delta t}}$ and $m = 1$.

Call options have a $\max(S_T - K, 0)$ payment, whereas put options have a $\max(K - S_T, 0)$ payoff. Using the backward induction process, where i denotes the time and j denotes the space, we get

$$C_{i,j} = e^{-r\delta t}[P_u C_{i+1,j+1} + P_m C_{i+1,j} + P_d C_{i+1,j+1}]$$

The reverse recursion for the American type looks like this [38]:

For call:

$$C_{i,j} = \max(S_{i,j} - K, e^{-r\delta t}[P_u C_{i+1,j+1} + P_m C_{i+1,j} + P_d C_{i+1,j+1}])$$

For put:

$$P_{i,j} = \max(K - S_{i,j}, e^{-r\delta t}[P_u C_{i+1,j+1} + P_m C_{i+1,j} + P_d C_{i+1,j+1}])$$

E. Finite Difference Method (FDM)

By estimating variables using finite differences, FDM is a computational approach for computing differential equations. FDM is technologically practical and essential for solving PDEs and pricing issues, providing a generic mathematical approach. The differential problem is transformed into a sequence of difference equations, which are then solved repeatedly. The most frequently used FDM for solving the Black-Scholes PDE is the:

1. Explicit Method
2. Implicit Method
3. Crank Nicolson Method

Now we discrete equation (1) concerning time t and the asset's fundamental value S. Convert the (t, S) plane into a suitably large number of points or mesh and accurately estimate the infinitesimal measures Δt and ΔS with tiny, defined, limited steps. Create an array of $N+1$ evenly divided grid points $t_0, ..., t_N$ to divide the periodic derivative using $t_{i+1} - t_i = \Delta t = \frac{T}{N}$. Following the same processes can calculate the asset's underlying price as follows: $S_{j+1} - S_j = \Delta S = \frac{S_{\max}}{M}$

This results in a rectangular area with sides on the (t, S) plane as $(0, S_{\max})$ and $(0, T)$. We may estimate the solution at discontinuous places using the grid coordinates (i and j). At time step t_i the stock price is S_j, and the designate price of the derivative as

$$f_{i,j} = f(i\Delta t, j\Delta S) = f(t_i, S_j) = f(t, S)$$

here i and j are discontinuous increases until maturity and share value, respectively. [30], [39]

To solve Equation (1) at a given time $i\Delta t$, we must first find the option values at:

- The upper border
- The lower border
- The starting values at option maturity.

The PDE on sector $S \in [0, \infty]$ participates in the European and American put. This poses a challenge. This field needs to be represented by a limited number of elements. Limiting the area to $S \in [0, M]$, where M is an appropriately large number, is a viable workaround [39].

Consider,

Time: $0, \Delta t, 2\Delta t, ..., N\Delta t;$ where $N\Delta t = T$

Price: $0, \Delta S, 2\Delta S, ..., M\Delta S;$ where $M\Delta S = S_{max}$

Let j=M at upper border so that $M\Delta S = S_{\max}$.

Now for European put:

$f_{i,M} = 0, \quad i = 0, 1, ..., N$

And for American put:

$f_{i,M} = 0, \quad i = 0, 1, ..., N$

Since $j = 0$ at lower limit, so $j\Delta S$ is zero and the payoff for the European put is:

$f_{i,0} = Ke^{-r(N-i)\Delta t}, \; i = 0, 1, ..., N$

And for the value of the American put option is:

$f_{i,0} = K, \qquad i = 0, 1, ..., N$

When $i = N$, the European put is:

$f_{N,i} = \max(K - j\Delta S, 0), \; j = 0, 1, ..., M$

And the American put is:

$f_{N,i} = \max(K - j\Delta S, 0), \; j = 0, 1, ..., M$

The aforementioned equations give the European and American put option values along the three grid conditions, where $S = S_{\max}$, $S = 0$ and $t = T$.

TABLE I
BOUNDARY CONDITIONS FOR PUT OPTION

Boundary	European Put Option	American Put Option
$t = T$	$f_{N,i} = \max(K - j\Delta S, 0)$	$f_{N,i} = \max(K - j\Delta S, 0)$
$S = S_{\max}$	$f_{i,M} = 0$	$f_{i,M} = 0$
$S = 0$	$f_{i,0} = Ke^{-r(N-i)\Delta t}$	$f_{i,0} = K$

Discretized by FDM scheme:

Substitute the partial derivative that appears in the partial differential equation by estimations relying on Taylor series approximations of the function at the sites of interest in the finite difference approach [26]. Expanding $f(t, S + \Delta S)$, $f(t, S - \Delta S)$, $f(t + \Delta t, S)$ and $f(t - \Delta t, S)$ in Taylor series so the forward and backward difference formed respectively as [39], [40]

$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j}}{\Delta S} \qquad (19)$$

$$\frac{\partial f}{\partial S} = \frac{f_{i,j} - f_{i,j-1}}{\Delta S} \qquad (20)$$

The central difference is obtained through the first-order partial derivative.

$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} \qquad (21)$$

Second derivatives terms can be estimated using the symmetric central difference technique. This may be calculated as

$$\frac{\partial^2 f}{\partial s^2} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta S^2} \quad (22)$$

$$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t} \quad (23)$$

$$\frac{\partial f}{\partial t} = \frac{f_{i,j} - f_{i-1,j}}{\Delta t} \quad (24)$$

Substituting Equations (21), (22), and (24) into the BSPDE (1) and noting that $S = j\Delta S$ and rearranging terms, we obtain the Implicit Scheme,

$a_j^* f_{i,j-1} + b_j^* f_{i,j} + c_j^* f_{i,j+1} = f_{i-1,j}$ for $j = 1,2,\ldots\ldots,M-1$ and $i = N,\ldots\ldots\ldots,1$

where

$a_j^* = \frac{1}{2}(\sigma^2 j^2 - rj)\Delta t;$

$b_j^* = 1 - (\sigma^2 j^2 + r)\Delta t;$

$c_j^* = \frac{1}{2}(\sigma^2 j^2 + rj)\Delta t;$

Similarly, after substituting Equations (21), (22), and (24) into the BSPDE (1) we get the Explicit Scheme,

$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j}$ for $j = 1,2,\ldots\ldots,M-1$ and $i = 0,1,\ldots\ldots,N-1$

where

$a_j = \frac{1}{2}(rj - \sigma^2 j^2)\Delta t;$

$b_j = 1 + (\sigma^2 j^2 + r)\Delta t;$

$c_j = -\frac{1}{2}(\sigma^2 j^2 + rj)\Delta t;$

The purpose now is to discretize the BSPDE (1). As a result, central estimation can be employed for $\frac{\partial f}{\partial t}$ at the point $f_{i-\frac{1}{2},j}$:

$$\frac{\partial f_{i-\frac{1}{2},j}}{\partial t} = \frac{f_{i,j} - f_{i-1,j}}{\Delta t} + O(\Delta t^2)$$

By using a central approximation for $\frac{\partial f}{\partial t}$ at the point $f_{i-\frac{1}{2},j}$:

$$\frac{\partial f_{i-\frac{1}{2},j}}{\partial S} = \frac{1}{2}\left[\frac{\partial f_{i-1,j}}{\partial S} + \frac{\partial f_{i,j}}{\partial S}\right]$$

$$= \frac{1}{2}\left[\frac{f_{i-1,j+1} - f_{i-1,j-1}}{2\Delta S} + \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S}\right] + O(\Delta S^2)$$

Now using a standard approximation for $\frac{\partial^2 f}{\partial S^2}$ at the point $f_{i-\frac{1}{2},j}$:

$$\frac{\partial^2 f_{i-\frac{1}{2},j}}{\partial S^2} = \frac{1}{2}\left[\frac{\partial^2 f_{i-1,j}}{\partial S^2} + \frac{\partial^2 f_{i,j}}{\partial S^2}\right]$$

$$= \frac{1}{2}\left[\frac{f_{i-1,j+1} - 2f_{i-1,j} + f_{i-1,j-1}}{\Delta S^2} + \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta S^2}\right] + O(\Delta S^2)$$

Replacing these values into equation (1) yields

$$-a_j^1 f_{i-1,j-1} + (1-b_j^1) f_{i-1,j} - c_j^1 f_{i-1,j+1} = a_j^1 f_{i,j-1} + (1-b_j^1) f_{i,j} - c_j^1 f_{i,j+1}$$

for $j = 1,2,\ldots\ldots,M-1$ and $i = N,\ldots\ldots\ldots,1$

where

$a_j^1 = \frac{\Delta t}{4}(\sigma^2 j^2 - rj);$

$b_j^1 = -\frac{\Delta t}{4}(\sigma^2 j^2 + r);$

$c_j^1 = \frac{\Delta t}{4}(\sigma^2 j^2 + rj);$

III. Numerical Results and Discussions

This article computes the American and a European put option value on a non-dividend paying stock. And the data sets considered here is taken from [40]:

Data Set:

The following data set is collected from [40].

$\{S_0 = \$69, E = \$70, r = 0.05, \sigma = 0.35, T = 0.5\}$

Where,

$S_0 =$ Stock Price

$E =$ Strike Price

$r =$ Risk Free Interest Rate

$\sigma =$ Volatility

$T =$ Maturity Time

Table II shows a comparison between [40] and my work which shows a good agreement.

TABLE II
Comparison between Debnath and Hossain [40] and present work

	European put option [40]	European put option [Present work]	% of error
BSM	6.4014	6.401	0.0000625
Implicit FDM	6.3926	6.392	0.0000939
Crank-Nicolson FDM	6.4012	6.420	0.0029283

Table III gives the exact price of the taken data set evaluated for the European and American put options using different methods.

TABLE III
Numerical values for data set in different methods

	BOPM	TOPM	MC	BSM	FDM		
					Explicit	Implicit	Crank Nicolson
European Put Option	6.401	6.402	6.105	6.401	6.402	6.392	6.420
American Put Option	6.576	6.577	--	--	6.579	6.559	6.574

Table IV shows the American put option price variation concerning the changes in interest rate (r), volatility (σ) the strike price (E), and maturity time (T) for Binomial Option Price Model (BOPM), Trinomial Option Price Model (TOPM), Explicit FDM, Implicit FDM and Crank Nicolson FDM.

TABLE IV
American put option

Affecting Factor	FactorsValue	BOPM	TOPM	FDM		
				Explicit	Implicit	Crank

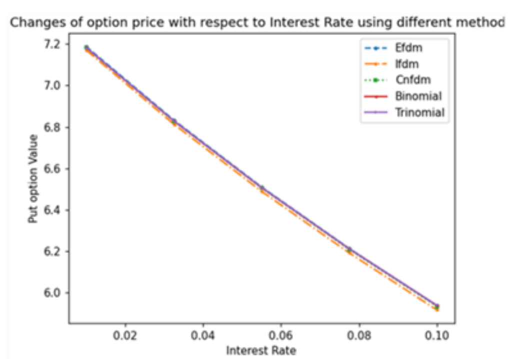
				t	t	Nicolson
Interest rate (r)	0.01	7.18175	7.18287	7.18739	7.17031	7.18175
	0.0325	6.82791	6.82883	6.8324	6.81274	6.82663
	0.055	6.50685	6.50761	6.50997	6.48914	6.50446
	0.0775	6.2115	6.21213	6.2133	6.19184	6.20809
	0.1	5.93801	5.93852	5.93778	5.91671	5.9336
Volatility (σ) (%)	29	5.42905	5.42944	5.43218	5.41343	5.49636
	31	5.85849	5.85962	5.86239	5.84272	5.89545
	34	6.28864	6.28982	6.29255	6.2722	6.38673
	36	6.71938	6.71995	6.72259	6.70183	6.79714
	38	7.15	7.14993	7.1524	7.13165	7.24768
Strike price (E)	60	2.40146	2.40128	2.40126	2.40852	2.4166
	70	6.50581	6.50659	6.50026	6.5586	6.57365
	80	13.07099	13.07106	13.07127	13.06154	13.07912
	90	21.41651	21.41642	21.40985	21.41241	21.42072
	100	31.0325	31.0735	31.0201	31.0551	31.3101
Maturity time (T)	0.5	6.57581	6.57659	6.57926	6.5586	6.59735
	0.88	8.21361	8.21137	8.21146	8.19609	8.71618
	1.25	9.41601	9.41384	9.40988	9.43247	9.55741
	1.625	10.37394	10.37185	10.36378	10.4683	10.49422
	2.0	11.17074	11.16872	11.15751	11.38569	11.41189

Changing values for different affecting factors are shown in Table V for European put option.

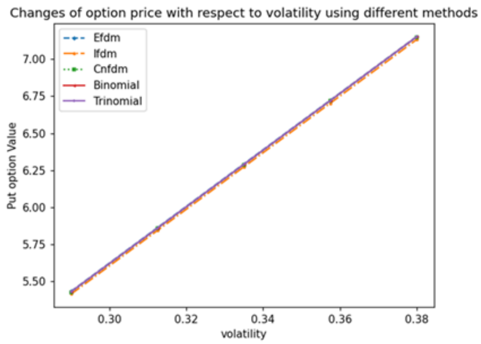
TABLE V
EUROPEAN PUT OPTION

Affecting Factor	Factors Value	BOPM	TOPM	MC	BSM	FDM		
						Explicit	Implicit	Crank Nicolson
Interest rate (r)	0.01	7.15692	7.15819	6.68646	7.15762	7.16318	7.14836	7.16131
	0.0325	6.72425	6.72553	6.72016	6.72496	6.73105	6.71568	6.73435
	0.055	6.31103	6.31157	5.79173	6.31102	6.31742	6.30182	6.32782
	0.0775	5.91476	5.91603	5.96404	5.91549	5.922	5.90647	5.93767
	0.1	5.53728	5.53856	5.37936	5.53804	5.54446	5.52928	5.56736
Volatility (σ) (%)	0.29	5.24809	5.24908	5.09038	5.24843	5.25549	5.24062	5.25299
	0.3125	5.67987	5.68178	5.29039	5.68106	5.68797	5.67262	5.78952
	0.335	6.11223	6.11404	6.05827	6.11338	6.12	6.1044	6.12747
	0.3575	6.54484	6.54584	6.18295	6.54536	6.55155	6.53606	6.56701
	0.38	6.97698	6.97717	7.12777	6.97695	6.98263	6.96783	7.40873
Strike price (E)	60	2.36943	2.37061	2.10998	2.37012	2.3636	2.36576	2.38874
	68	5.4077	5.40912	5.25888	5.40853	5.41481	5.39931	5.49331
	76	9.90085	9.90118	9.78186	9.90077	9.89589	9.89418	9.92436
	84	15.6021	15.6012	15.1895	15.6011	15.6072	15.6033	15.7211
	92	22.1632	22.1620	22.0462	22.1628	22.1675	22.1814	22.1974
Maturity time (T)	0.5	6.40069	6.40196	6.39702	6.40141	6.40775	6.39218	6.41874
	0.875	7.88074	7.87832	8.03178	7.8793	7.88312	7.88421	8.00884
	1.25	8.91159	8.90943	9.2149	8.90927	8.91056	8.995	9.26116
	1.625	9.68785	9.68594	9.75416	9.68515	9.68418	9.94779	10.34041
	2.0	10.2952	10.2935	10.2558	10.2923	10.2893	10.8276	10.5776

Figure 1 shows the changes of American put option against the interest rate, volatility, strike price, and maturity time with the Explicit, Implicit, Crank Nicolson, Binomial tree, and Trinomial tree methods that are also quantitatively presented in Table IV. In Fig. 1(a), the price of a put option falls as interest rates rise. Rho determines how sensitive an option is with fluctuations in the risk-free interest rate. Put options have a negative Rho; thus, when interest rates rise, the price falls significantly. In Fig. 1(b), the put option prices go up as a result of growth in volatility. The rise in stock volatility enhances the value of both the call and put options. So, it is seen that only volatility influences put options similarly. In Fig. 1(c), the put price rises almost exponentially as the strike price rises. A rising strike price gives an increasing option price. It is clear from Fig. 1(d) that as the maturity time grows, so does the put option price. An option's temporal value decreases as it goes more out of the money; its underlying value is zero. An option's time value is at its highest point when it is in the money.



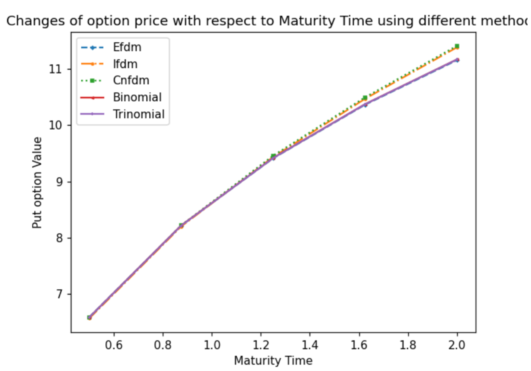
(a) American put option value against interest rate



(b) American put option value against volatility

(c) American put option against strike price

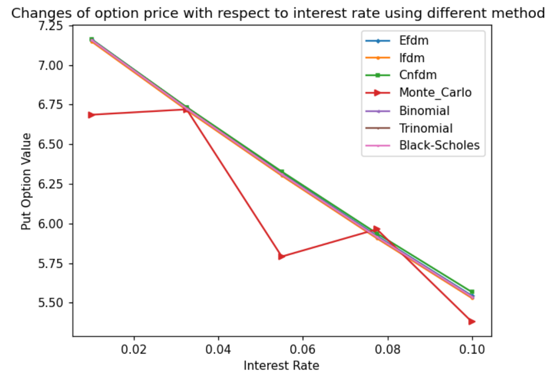
.



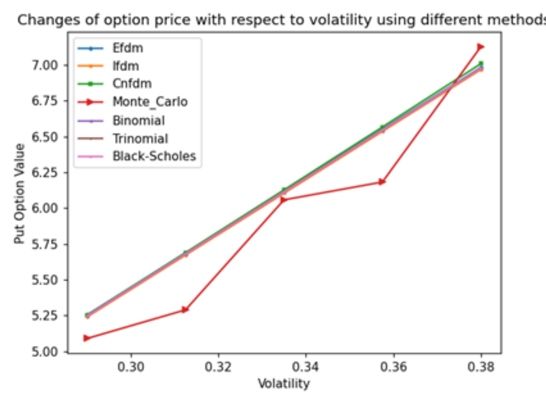
(d) American put option against maturity

Fig.1. American put option pricing concerning the changes in interest rate, volatility, strike price, and maturity time.

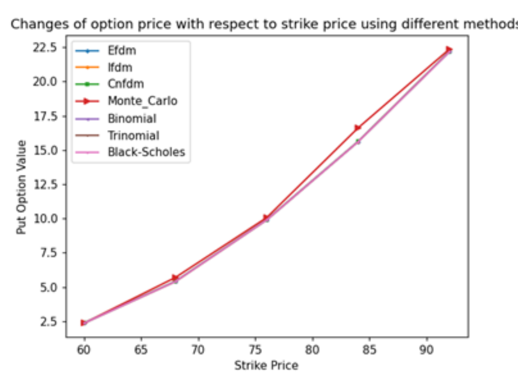
A graphical representation of Table V is shown in Fig. 2. Because of the negative rho, the put option value in Fig. 2(a) drops as the interest rate rises. Figure 2(b) demonstrates how volatility boosts the put option's price. It is clear from Fig. 2(c) that when the strike price rises, the put price does as well. The solutions provided by the seven approaches given are nearly identical. Figure 2(d) shows the put options towards the maturity time. When the time value goes up, the option price also rises.



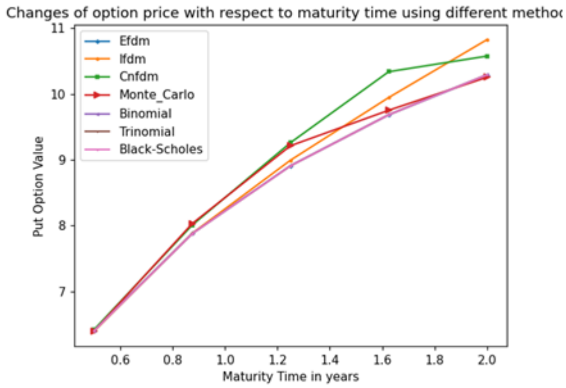
(a) European put option value against interest rate



(b) European put option value against volatility



(c) European put option value against strike price



(d) European put option value against maturity

Fig. 2. European put option pricing concerning the changes in interest rate, volatility, strike price, and maturity time.

IV. CONCLUSION

Five basic numerical techniques in financial mathematics are summarized in this study. Applied techniques include the Black-Scholes-Merton, Monte Carlo, Binomial, Trinomial, and Finite Difference approaches. The Monte Carlo and Black Scholes methods work well with European price options, whereas all other methods suit European and American options. The Monte Carlo approach is adaptable for dealing with high-dimensional financial problems. FDM is more accurate and is suitable for option pricing. Crank Nicolson FDM gives the best result among the Explicit, Implicit, and Crank Nicolson methods.

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